

Discovering

the Riemann Zeta Function

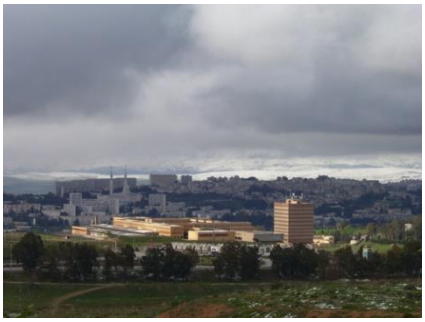
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The book still being updated, any mistakes occurring here are mine.

Preface

Interested in number theory, I was one day at the local municipal Library reading the book of English mathematician William J. **Ellison** untitled *"Les nombres Premiers"*(Hermann 1975). I have begun from the first page thinking that it would not very difficult since the book treated only integers!, but the introduction of real and complex variables disoriented me, not understanding much the developments until I arrived to this passage

***Riemann** has conjectured that $\zeta(z)$ does not vanish in any point of the half-plan $\Re(z) > \frac{1}{2}$, this conjecture, known under the name of **Riemann** hypothesis is far to be solved and constitutes one of the most famous mathematical problems.*

The passage has impressed me much, although I have already heard of the name of **Riemann** but never of a "hypothesis" linked to him. Then I immediately went into the Internet to know the nature of this such difficult "*Hypothesis*" and why not try to prove it ! and I discovered that it says

*All non-trivial roots of the function ζ of **Riemann** are of the form $\frac{1}{2} + it$*

Not understanding too much this sentence either, I searched again in the Internet for **Riemann**'s article, began to study it more closely, my researches permeated me to know **Riemann** himself, his work , and above all his article of 1859, which gave great

impetus to the newly created domain named *analytic numbers theory*. Two questions bothered me a lot

*How **Riemann** discovered that ζ has other roots than the trivials? And how did he suspect a relation between the prime numbers and these non-trivial roots?*

Since the article of **Riemann** was very condensed and difficult, I began the work to simplify it for myself first, then for beginner students in the domain of analytic theory of numbers. And especially I tried to explain the implied tools of complex analysis, the language in which the article was written.

Riemann's article of 1859 , entitled *" On the number of prime numbers under a given quantity "* , marked a new epoch in number theory and in mathematics in general, and according to the words of Harold **Edwards**:

*No mathematical work is more clearly a classic than **Riemann**'s memoir "Ueber die Anzahl der Primzahlen unter einner gegebenen Grösse" published in 1859.*

Mathematicians have seen with the memoir of **Riemann** the birth of *the **Riemann** zeta function* , which has become one of the most famous functions in mathematics, a lot of problems in analytic number theory have for origin the study of this obscure function.

Books on this subject abound, but are difficult enough and need a good knowledge of complex analysis, which make them not suitable at all for beginners as a first touch with this difficult branch of mathematics. To the known books of : **Titchmarsh** (1930 and 1951), **Edwards**(1974) , **Ivic**(1985) , **Paterson**(1989), **Karatsuba**(1995), the present book is considered as an introduction to those more specialized references.

I will explain **Riemann**'s reasoning step by step, above all I will insist well on the key results of complex analysis that will be necessary. The theorems are stated in their historical contexts by giving, as far as possible, motivations in order to lead to his final result. In order not to burden the text, I have not given all the proofs of the theorems used by **Riemann**; they find their places in specialized books given in the bibliography. My aim will be to appease the aridity of the text to the reader. I also

from time to time make diversions from **Riemann**'s text to mention some modern methods and developments forged by specialists to attack the **Riemann** hypothesis.

For a simpler translation, I have done a comparison between 4 translations of **Riemann**'s article contained in:

- *Oeuvre de Riemann*, translated into French by **Laugel** (Gabay, 1990).
- *Works of Riemann*, translated into English (Kendrick Press, 2004).
- English translation by Harold **Edwards**(1974).
- English translation by David **Wilkins**, university of Dublin(1998)

Dirichlet has introduced the real variable $\boldsymbol{s}$ in his work of 1837, then used unfortunately by **Riemann** for complex variable z, although he used the letter z for complex variables in his other articles. I have used the letters $z, w, \ldots$ for complex variables following modern usage of mathematicians.

To convey the reader to participate actively and understand more the subject, every chapter ends with exercises in real and complex calculus. This is a preparative text (I originally wrote it in French, I am now progressively translating it into English), I would be very happy if it will be at the good hope of all readers who are interested in the **Riemann** hypothesis.

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*« **Quand j'étais jeune, j'ai eu l'espoir de démontrer l'hypothèse de Riemann. Quand je suis devenu un peu plus vieux, j'ai eu encore l'espoir de pouvoir lire et comprendre une démonstration de l'hypothèse de Riemann. Mais maintenant, je me contente de savoir qu'il en existe une** ».*

André **Weil** (1906-1998)

*« **Ignorance prevents me from entering into a discussion of these functions and what is known about them** ».*

Harold M. **Edwards** (1936 – 2020)

I) Riemann′s article

On the quantity of primes less than a given quantity



Definition of the ζ function

I believe that I can best express my gratitude for the honor which the Academy has bestowed on me in admitting me as one of its correspondents…

The *Academy of Sciences* in Berlin chose Bernhard **Riemann** (1826-1866) as one of its correspondents on 11 August 1859. He has just became a full Professor at the university of Göttingen after the death of his friend and teacher Gustave **Dirichlet** (1805-1859) who himself replaced **Gauss**(1777-1849) in 1855 at the same university.

Riemann then, was already known as a talented mathematician within German mathematical community, he started his mathematical studies in 1846 as a student of **Gauss** and **Stern**(1807-1894) at the university of Göttingen, then he visited the University of Berlin in the years 1847-1849 where he studied under **Dirichlet**, **Steiner**(1796-1863) , **Jacobi** (1804-1851) and **Eisenstein**(1823-1852). In 1849, he returned to Göttingen, where he assisted the physicist Wilhelm **Weber**(1804-1891)

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until 1851 , the year he obtained his PhD. We can retrace **Riemann**'s work before 1859 as follows:

1851: *General Theory of functions of a complex variable (Göttingen),* which was his *Doctoral dissertation*, and in which appeared the concept of *Riemann surface*.
1854 : *On the laws of the distribution of voltage electricity*…(*Göttingen)*
1854 : *On the hypotheses which lie at the bases of Geometry (Göttingen)*, read to **Gauss**, it threw the foundation of modern differential geometry.
1854 : *On the possibilities to represent a function by a trigonometric series*, which was his *habilitation* (*Habilitationsvortrag*) . It earned him a *Privatdozent* (the right to teach at the university with a salary), and in it he introduced his famous definition of the definite integral known as *Riemann Integral*.
1855 : *On the theory of **Nobili**'s color rings (in physic-chemistry*), it was on the flow of current in a body and **Riemann** treat the problem by calculus. Leopoldo **Nobili**(1784-1835) was an Italian physicists who discovered these rings in a solution of cooper by electric charges.
1855: *Theory of Abelian functions*, this was treated by using complex variables. It appeared in **Crelle**'s Journal(1857), a precursor work to the creation of modern algebraic geometry,
1857: *Contribution to the theory of functions representable by a **Gauss** series* $F(\alpha, \beta, \gamma, x)$. **Gauss** introduced series in 1812.
1857 : *Two general theorems on the ordinary linear differentials equations with algebraic coefficients*, in this year he became an associate Professor.
1858 : *A contribution to electrodynamics*.

This nomination was under the initiative of Karl **Weierstrass** (1815-1895) and Ernest **Kummer**(1810-1893). The occasion of visiting the Academy in 1859 with his friend **Dedekind** (1831-1916), encouraged him to unveil a part of his researches, although not yet finished, on the distribution and density of prime numbers which is the subject of this book. This was his only paper in number theory; a subject very enjoyed by several of his friends and his teachers, among them **Gauss**, **Jacobi** and **Dirichlet**.

by immediately availing myself on the privilege this entails, to communicate an investigation concerning the frequency of primes. It is a subject that, because of the interest shown in it by Gauss…

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Context of the paper

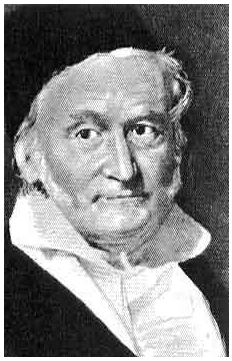
The *frequency* or *density* of primes is a problem about the distribution of prime numbers. It means:

How prime numbers are distributed inside of the set of integers $\mathbb{N}$?

a problem about which some results were already known at the time:

– There are infinitely many prime numbers in the sequence of integers: 0, 1, 2, … , a result which has been obtained by **Euclid**, in modern notation, let the symbol $\pi(x)$ be the number of primes less than a real quantity x, or in mathematical notation

$$\pi(x) = \sum_{p \leq x} 1$$



Then **Euclid** result is to be understood as

$$\lim_{x \to +\infty} \pi(x) = +\infty$$

– The series $\sum_p \frac{1}{p}$ diverges, which implies that the sequence of prime numbers is denser in $\mathbb{N}$ than the sequence of the squares n^2 , as the series $\sum_n \frac{1}{n^2}$ converges.

– There exits at least a prime between n and $2n$, a propriety conjectured by French mathematician Joseph **Bertrand**(1822-1900) in 1845 and proved[1] by Russian mathematician Pafnuty **Tchebychev**(1821-1894) in 1852.

Gauss, teacher of **Riemann** at the university of Göttingen, was interested in the subject during his early youth, and he has made a conjecture:

[1] *Mémoire sur les nombres premiers*", in his *Collected Work* ($\boldsymbol{I/p.51}$).

In a letter to German astronomer Johann **Encke** (1791-1865) in 1849, by reposing on extensive calculations he had effectuated since 1791 or 1972, **Gauss** states that

$$\pi(x) \sim \int_2^x \frac{1}{\log(t)} dt$$

where $\int_2^x \frac{1}{\log(t)} dt$ is known as the *logarithm integral* and noted $Li(x)$, it is a positive real number that represents the area between the curve $f(t) = \frac{1}{\log(t)}$ and the real abscissa bounded by 2 and a given x.

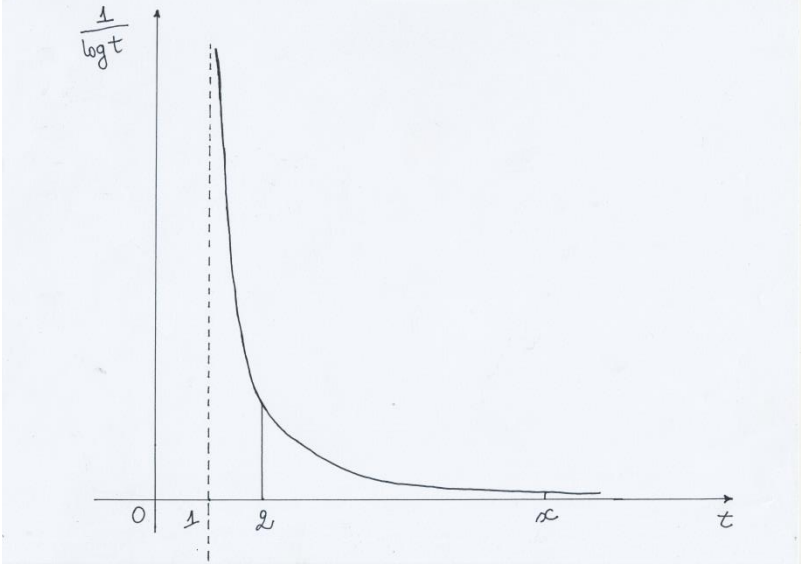


Fig.1 The function $Li(x)$ as the area $\int_2^x \frac{1}{\log(t)} dt$

The antiderivative of $\frac{1}{\log(t)}$ is not an elementary function but it is easy to show that $\int_2^x \frac{1}{\log(t)} dt$ diverges when x tends to $+\infty$: since for $t \geq 2$

$$\frac{1}{logt} > \frac{1}{tlogt} > 0$$

Then

$$\int_2^x \frac{1}{\log(t)} dt > \int_2^x \frac{1}{t\log(t)} dt \sim loglog(x)$$

which tends to $+\infty$ when x tends to $+\infty$.

Gauss's estimation means

$$\pi(x) = \int_2^x \frac{1}{\log(t)} dt + Error(x)$$

where $Error(x)$ is a term that depends on the variable x and tends to 0 when x tends to $+\infty$, or otherwise

$$\lim_{x \to +\infty} \left| \pi(x) - \int_2^x \frac{1}{\log(t)} dt \right| = \lim_{x \to +\infty} |Error(x)| = 0$$

from integral calculus

$$F(x) = \int f(x) dx$$

or $F' = f$, so

$$d\pi = (\frac{1}{\log(x)} - \frac{1}{\log(2)})dx$$

here f is seen as a *mean value* (or *density*) of $F(x)$, roughly

$$\frac{\Delta F(x)}{\Delta x} \sim f(x)$$

then **Gauss** means that the *density* of prime numbers in $\mathbb{N}$ is $\frac{1}{\log(t)}$, i.e.

$$\frac{\pi(x)}{x} \sim \frac{1}{\log(x)}$$

Otherwise

$$\pi(x) \sim \frac{x}{\log(x)}$$

So the is that is the probability that x is a prime is $\frac{1}{\log(x)}$ as

$$\frac{\pi(x)}{x} = \frac{number\ of\ prime\ cases}{number\ of\ all\ cases} = \frac{1}{log(x)}$$

And gauss thought that

$$\pi(x) \sim \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{x}$$

The reader notes also that when integrating by parts we find that $\int_2^x \frac{1}{\log(t)} dt$ is not *very different* from $\frac{x}{\log(x)}$ for

$$\int_2^x \frac{1}{\log(t)}dt = \frac{x}{\log(x)} - log2 + \int_2^x \frac{1}{\log(t)^2}dt \sim \frac{x}{\log(x)}$$

then the observation de **Gauss** means that

$$\pi(x) \sim \int_2^x \frac{1}{\log(t)}dt \sim \frac{x}{\log(x)}$$

In the same current of ideas, and in his book $Théorie\ des\ Nombres$(1798-1820) which **Riemann** had read when he was at the Gymnasium(*high school*[2]), French mathematician **Legendre** (1752-1833) affirmed that

$$\pi(x) \sim \frac{x}{\log(x) - A}$$

where $A = 1.08366$, however **Legendre**'s constant A has no sense at all and must be dropped out because for any finite real number A_i

$$\lim_{x\to+\infty} \frac{x}{logx - A_1} = \lim_{x\to+\infty} \frac{x}{logx - A_2} \ldots . = \lim_{x\to+\infty} \frac{x}{logx}$$

Edwards remarked:

Legendre *does not specify the sense in which the (above) approximation is to be understood, but if it is interpreted in the usual sense of "the relative error approaches zeros as* $x \to +\infty$ *", then the value of* A *is irrelevant because* $\frac{x}{logx-A} \sim \frac{x}{logx-B}$ *for any two numbers* A, B *(because the ratio is* $logx - A$ *over* $logx - B$ *which approaches 1) and hence if(the above approximation) is true for one value of* A*, it must be true for all values of* A*(because* $\sim$ *is transitive). Therefore if* ***Legendre****'s value* $A = 1.08366$ *has any significance, it must lie in some other interpretation of the (above) approximation…*

[2] In the town of Lüneburg , south east of Hamburg.

Speculating on the conjecture of **Legendre**, **Tchebychev** showed[3] in 1848 that

$$0{,}89 \times \int_2^x \frac{1}{\log(t)}\,dt < \pi(x) < 1{,}11 \times \int_2^x \frac{1}{\log(t)}\,dt$$

namely he showed that the infinite sum

$$\sum_{x=2}^{x=+\infty} [\pi(x+1) - \pi(x) - \frac{1}{\log(x)}] \frac{log^n(x)}{x^{1+\alpha}}$$

tends to the finite limit α, and that

$$\int_2^x \frac{1}{\log(t)}\,dt - \frac{\alpha x}{log^n(x)} < \pi(x) < \int_2^x \frac{1}{\log(t)}\,dt + \frac{\alpha x}{log^n(x)}$$

This means that the estimation of **Gauss** was not bad, but what remained was to prove the limit

$$\lim_{x\to+\infty} \frac{1}{\pi(x)} \int_2^x \frac{1}{\log(t)}\,dt$$

is 1 , or otherwise

$$\lim_{x\to+\infty} \frac{\pi(x) \times logx}{x} = 1$$

[3] "*Sur la fonction qui détermine la totalité des nombres premiers inferieurs à une limite donnée*", in his *Collected Work* $(\boldsymbol{I/p.29})$.

which **Tchebychev** could not prove.

Riemann was acquainted with **Tchebychev**'s work, and he used some of his ideas as we will see later.

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…and Dirichlet for many years , seems to me not unworthy of such a communication.



Another influence on **Riemann** comes from his teacher Lejeune **Dirichlet** who has proved for the first time[4] in 1837 that there are infinitely many prime numbers in every arithmetic progressions of the form $ak + b$, where a and b are without common factors otherwise: $(a, b) = 1$.
In his proof, **Dirichlet** used the **Euler** identity

$$\sum_{n=1}^{\infty} \frac{1}{n^x} = \prod_{p} (1 - \frac{1}{p^x})^{-1}$$

this celebrated identity establishes a curious relation between in one part naturel numbers and in another part prime numbers. The equality was considered by **Euler** for the first time in 1737.

Dirichlet gave a proof of this equation for a real number x, and introduced new a function χ that *characterizes* primes in the sequence $ak + b$

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^x} = \prod_{p} (1 - \frac{\chi(p)}{p^x})^{-1}$$

[4] **Legendre** gave a false proof of this result in his book $Théorie\ des\ Nombres$.

showing that

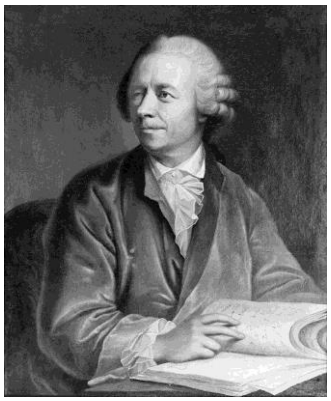
$$\lim_{x\to 1}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^x}=+\infty$$

Thus he introduced new methods, named *analytic* , by using the real variable x to solve a problem in the theory of numbers, he is sometimes considered the founder of *analytic number theory*.

Euler wrote the above identity for any integer $x=k$ and without a proof

$$\sum_{n=1}^{\infty}\frac{1}{n^k}=\prod_{p}(1-\frac{1}{p^k})^{-1}$$

But he gave an interesting formula pour $x=2k$



$$\sum_{n=1}^{\infty}\frac{1}{n^{2k}}=\frac{(-1)^{k-1}B_{2k}(2\pi)^{2k}}{2(2k)\,!}$$

where B_{2n} are named ***Bernoulli*** *numbers*. In particular, he solved in 1734 the *Basel problem ,* namely he calculated the sum

$$\sum_{n=1}^{+\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

An open problem posed by the Italian **Mengoli** (1626-1686) in 1644-50 and had resisted to all the attempts by the **Bernoullis** who lived in $Basel$ at the time (so the name).

Euler also touched the problem of the density of prime numbers, writing what is looked like[5]

$$\sum_{p\leq x}\frac{1}{p} \sim log(log(x))$$

which is equivalent to say that

$$\sum_{p\leq x}\frac{1}{p} \sim \int_{e}^{x}\frac{1}{\mathrm{tlog}(t)}dt$$

And the integral

$$\int_{e}^{x}\frac{1}{\mathrm{tlog}(t)}dt$$

diverges. Of course this is only an approximation, for **Euler** wrote $loglog(\infty)$, and we do not know the how big the sign ∞ is, but German mathematician **Mertens** (1840-1927) proved that really $\sum_{p\leq x}\frac{1}{p}\sim loglogx$ by proving his known theorem

Martens' Theorem (1874):

We have

$$\lim_{x\to+\infty}(\sum_{p\leq x}\frac{1}{p} - loglogx) \approx 0.26$$

[5] See a proof that $\sum_{p=2}^{\infty}\frac{1}{p}$ diverges in **Apostol** p. 18.

See also **Edwards** (p.2) for an interpretation of **Euler**'s observation.

Function of a complex variable:

The work of **Dirichlet** has inspired much **Riemann** and we will see that he has pushed his teacher's ideas more ahead introducing new powerful methods by using both complex and **Fourier** analysis.

I will take as my starting point the observation of Euler that the product

$$\prod \frac{1}{1-\frac{1}{p^s}} = \sum \frac{1}{n^s}$$

when p takes all the primes, and n all the integers.

Riemann follows **Dirichlet** , and before him **Euler**, in considering the equality

$$\sum_{n=1}^{\infty} \frac{1}{n^z} = \prod_{p} (1-\frac{1}{p^z})^{-1}$$

Riemann was the first to consider in place of real x the complex number $z = \sigma + it$, where the number i (notation of **Euler** in 1777) verifies

$$i = \sqrt{-1} \qquad \text{and} \qquad i^2 = -1$$

By the identity

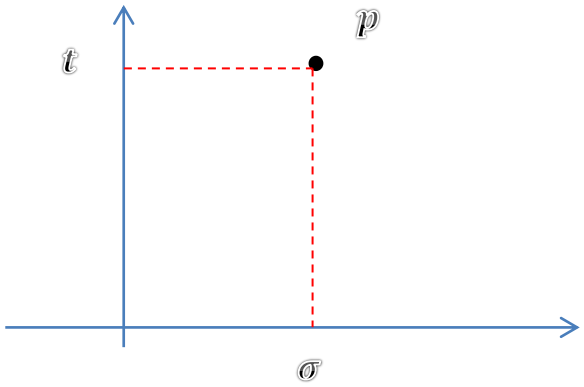
$$n^z = \cos(t\log n) + i\sin(t\log n$$

the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^z}$ is simply

$$\sum_{n=1}^{\infty} \frac{cos(t\log n)}{n^\sigma} - i\sum_{n=1}^{\infty} \frac{sin(t\log n)}{n^\sigma}$$

The function presented by the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^z}$ and at the same time by the product $\prod_p (1 - \frac{1}{p^z})^{-1}$ will be noted $\zeta(z)$, the zeta function.

The set of complex numbers $\mathbb{C}$ is what mathematicians call a *field* , which means that we can deal freely with the operation $(+,\ -,\ \times,\ \div)$ like in the set $\mathbb{R}$ expect the order, $\mathbb{C}$ is not an *ordered field* like $\mathbb{R}$: we cannot compare between two complex numbers z_1 and z_2.



A complex number $z = \sigma + it$ is a point p in the plane $\mathbb{R}^2$, and formerly mathematicians called it the *affix* of the point p when, as **Dieudonné** said : "*they still made a difference between algebra and geometry*".

In the real case, numbers are situated on a real line but in the complex case, numbers are situated on a plane, this why we need two parameters to characterize a complex number z :

1) The *module* $|z|$ the length or the distance from z to $0 = 0 + i0$ which is

$$|z| = \sqrt{\sigma^2 + t^2}$$

in the real case it is the absolute value $|x|$ the distance between x and 0.

2) *The argument* θ noted by $arg(z)$ (or $Arg(z).$) . It is the angle, in radian, between the segment $[0, z]$ and the positive real axis.

Thus we can write the complex number z from geometry in the *polar form*

$$z = \sigma + it = |z|e^{i\theta}$$

But **Euler** discovered in 1748 that

$$e^{i\theta} = \sum_{n\geq 0} \frac{(i\theta)^n}{n!} = \sum_{n\geq 0} \frac{(i\theta)^{2n}}{2n!}(-1)^n + \sum_{n\geq 0} \frac{(i\theta)^{2n+1}}{(2n+1)!}(-1)^n = \cos(\theta) + isin(\theta)$$

which gives

$$z = |z|[cos(\theta) + isin(\theta)]$$

the argument θ is not unique, in fact there are infinitely many because both cos and sin are periodic functions :

$$cos(\theta) = cos(\theta + 2n\pi), \qquad sin(\theta) = sin(\theta + 2n\pi)$$

We choose the first argument, that verifies

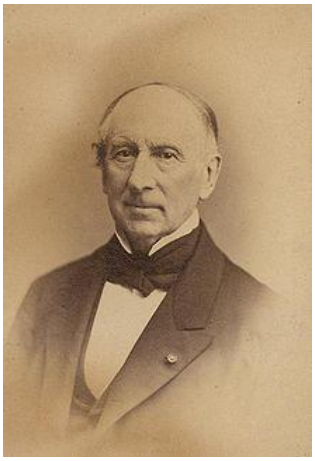
$$-\pi < \theta \leq \pi \qquad \text{or the same} \qquad 0 \leq \theta < 2\pi$$

it is called the *principal value* of θ . We have for example

$$arg(z_1 z_2) = arg(z_1) + \arg(z_2) \ \ mod(2\pi)$$

so we rather write

$$z = \sigma + it = |z| e^{i(\theta + 2n\pi)}$$



If the reader is frightened by complex numbers, he has seen after the appearance of cos and sin , that a complex number z is still linked to the geometry of the plane and to the real world by very obscure relations such as

$$i^{-2i} = e^{\pi}$$

The theory of complex variables has seen its early developments with **Euler** who saw $f(z)$ as a function of reals : σ and t . He considered some elementary functions like

$$cos(z), \ \ log(z), \ \ e^z, \ \ arctan(z), \ldots$$

he defined the complex integral (1777) and used it to calculate real integrals and for other applications.

Then the theory saw all its glory between the hands of **Cauchy**, in his work on definite integrals in the complex plan (1814)

$$\int_{z_1}^{z_2} f(z)dz$$

he considered f as a function of the variable z, and not a function of σ and t.

Still for other purposes, a function $f(z)$ may be seen as an application between two real planes

$$\begin{matrix} f:(\sigma,t) & \longrightarrow & (u,v) \\ \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \end{matrix}$$

The complex function f is divided into two parts (σ,t) and (u,v)

$$f(\sigma+it)=u(\sigma,t)+iv(\sigma,t)$$

A function $f(z)$ is continuous at a point z iff u and v are both continuous at (σ,t). Functions of two real variables $f(\sigma,t)$ has some classical results like

Theorem (Schwarz)

A function f of two real variables σ and t is derivable at the point (σ_0,t_0) and both $\frac{\partial f}{\partial\sigma\partial t}$, $\frac{\partial f}{\partial t\partial\sigma}$ are continuous at (σ_0,t_0) then

$$\frac{\partial f}{\partial\sigma\partial t}=\frac{\partial f}{\partial t\partial\sigma}$$

but if one of the ∂f is not continuous, the result may be false , as the classical example due to **Peano**(1858-1932) of the function

$$f(\sigma,t)=t\sigma\frac{\sigma^2-t^2}{t^2+\sigma^2}$$

at the point $(0,0)$, where $\frac{\partial f}{\partial \sigma \partial t}(0,0) = -1$ and $\frac{\partial f}{\partial t \partial \sigma}(0,0) = 1$.

Holomorphic functions:

All complex analysis is about *holomorphicity*. Like a function of real variables we can derivate a function of complex varible $f(z)$. We say, as in the real case, that f is drivable at z if

$$\lim_{\Delta z \longrightarrow 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

exists and has a well determined unique value. We give now the exact definition of holomorphicity:

Definition

If a complex function f is derivable at a point z_0 , then it is *holomorphic* at z_0.

We also say that f is *regular.* The word *holomorphic* (from $holo$ 'entire, whole' and $morphus$ 'form') coined by French **Briot** $(1817-1882)$ and **Bouquet** $(1819-1885)$. The two men studied under **Cauchy**, they wrote in 1856 a book entitled : "*Studies of functions of imaginary variables*" , a book to which **Riemann** refers often in his lectures.

In the real case f is not derivable at a point x if f is not *defined* at x or the limit of the ratio is infinite or has two different limits

$$\lim_{\substack{\Delta x \to 0 \\ > \\ \Delta x \to 0}} \frac{f(x+\Delta x) - f(x)}{\Delta x} \neq \lim_{\substack{\Delta x \to 0 \\ < \\ \Delta x \to 0}} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Unlike the real case, the Δz has an infinity of ways to tend the $0 + i0$ since it is in $\mathbb{C}$ (= the plane $\mathbb{R}^2$) and we cannot study them all. We can reformulate the derivability of a function of complex variable by using conditions already known to **Euler** and **d'Alembert** , called *: the equations of* ***Cauchy-Riemann*** :

Theorem (Cauchy-Riemann)

A f is derivable at the point z_0 or the same (σ_0, t_0), *iff* it verifies the **Cauchy-Riemann** equations

$$\frac{\partial u}{\partial \sigma} = \frac{\partial v}{\partial t} \quad , \quad \frac{\partial u}{\partial t} = -\frac{\partial v}{\partial \sigma}$$

These conditions are very deep, because they are also equivalent to say that the function f is a function of the variable z and not of σ or t, as **Riemann** said in his PhD thesis (1851):

"Hence this condition is necessary and sufficient for $u + iv$ *to be a function of* $z = x + iy$ *."*

A function of two variables $u(\sigma, t)$ is differentiable at a point (σ, t) if the partial derivatives

$$\frac{\partial u}{\partial \sigma} \quad and \quad \frac{\partial u}{\partial t}$$

exist and are continuous(otherwise : u and v are *continuously differentiable*) at (σ_0, t_0). But for a holomorphic function this is always true as the French mathematician Edouard **Goursat** (1858-1936) proved later:

Theorem (1900)

If $f(z)$ is derivable at a point z_0 then $f'(z)$ is continuous at z_0.

That is both $\frac{\partial u}{\partial \sigma}, \frac{\partial u}{\partial t}, \frac{\partial v}{\partial \sigma}$ and $\frac{\partial v}{\partial t}$ are continuous at (σ_0, t_0)..

Any functions u and v that verify **Cauchy**-**Riemann** equations are called *harmonic* functions, they both are the solution of the **Laplace** equation, for by **Schwartz** theorem we can find:

$$\frac{\partial^2 f}{\partial \sigma^2} + \frac{\partial^2 f}{\partial t^2} = 0$$

A surprising propriety of functions of complex variable is that : if f is derivable one time at a point z_0 (i.e. holomorphic at z_0) then it is derivable infinity many times at z_0 i.e. for all orders as we will see later . This propriety is not true in the real case: the function of real variable $f(x) = x^{\frac{1}{3}}$ is derivable on $\mathbb{R}$ but its derivative $f(x) = \frac{1}{3}x^{-\frac{2}{3}}$ is derivable only on $\mathbb{R}^*$.

If a function f is holomorphic at z_0 then it can be represented by a ***Taylor*** *series* at the point z_0

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

as we will see later.

We can prove the **Cauchy-Riemann** condition as follows. Let $w = f(z)$ then: $\frac{dw}{dz} = f'(z)$, so:

$$\frac{dw}{dz} = \frac{d(u(x,y) + iv(x,y))}{d(x + iy)} = \frac{du(x,y)}{dx + idy} + i\frac{dv(x,y)}{dx + idy}$$

We know that the limit does not depend on the path , so when $dx = 0$ the limit gives

$$\frac{dw}{dz} = \frac{du(x,y)}{0 + idy} + i\frac{dv(x,y)}{0 + idy} = -i\frac{\partial u(x,y)}{dy} + \frac{\partial v(x,y)}{dy}$$

and likewise when $dy = 0$ we have

$$\frac{dw}{dz} = \frac{du(x,y)}{dx + i0} + i\frac{dv(x,y)}{dx + i0} = \frac{\partial u(x,y)}{\partial x} + i\frac{\partial v(x,y)}{\partial x}$$

Equating the two terms gives us

$$\frac{\partial u(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y} \qquad and \qquad \frac{\partial u(x,y)}{\partial y} = -\frac{\partial v(x,y)}{\partial x}$$

Exercises

1) find $\int logx.dx$.

1) calculate $\int \frac{dx}{\mathrm{xlog}(x)}$.

1) calculate the area limited by the curve of $f(x) = x^3$ and $[-1,1]$.

1) the same for $g(x) = |x^3|$, $h(x) = |x|^3$.

1) draw the graph of the function $\frac{1}{\log(t)}$ for $t \in]0, +\infty[$

1) find both

$$\lim_{\varepsilon \to 0} \int_0^{1-\varepsilon} \frac{dt}{\log(t)}$$

and

$$\lim_{\varepsilon\to 0}\int_{1+\varepsilon}^{2}\frac{dt}{\log(t)}$$

1) Show that $\lim_{x\to\infty}\frac{\int_2^x \frac{dt}{\log(t)}}{\frac{x}{\log(x)}}=1$

1) show that $\int_2^x \frac{1}{\log^2(t)}$ is too small compared to $\frac{x}{\log(x)}$.

1) show that

$$log(log(x))=\int_e^x \frac{1}{tlog(t)}dt$$

1) prove by another estimate that $\lim_{x\to+\infty}\int_2^x \frac{1}{\log(t)}dt=+\infty$.

1) show that the sum

$$\sum_{n\geq 2}^{\infty}\sum_{p}^{\infty}\frac{1}{n}p^{-n}$$

converges, but

$$\sum_{p}^{\infty}p^{-1}$$

diverges.

1) how can you make $\mathbb{C}$ an ordered set?

1)show that

$$\sqrt{1+\sqrt{-3}}+\sqrt{1-\sqrt{3}}=\sqrt{6}$$ (**Leibnitz** 1702)

1) write the following in the form $\sigma+it$:

i) $log(1-i)$

ii) $\sqrt{2+2i}$

iii) $\frac{1+i}{1-i}$

1) write the argument θ of z as a function of σ and t (which is not always one to one)

1) show that $Re(z) = \frac{z+\bar{z}}{2}$, $Im(z) = \frac{z-\bar{z}}{2i}$

1) if $z_2 \neq 0$, show that $\frac{z_1}{z_2} = |z|e^{i\theta}$, find z and θ.

1) prove that

i) $cos(\sigma + t) = cos(\sigma)\cos(t) - sin(\sigma)sin(t)$

ii) $sin(\sigma + t) = cos(\sigma)sin(t) + cos(t)sin(\sigma)$

iii) $cos^2(\sigma + t) + sin^2(\sigma + t) = 1$

1) show that

$$cos(x + \frac{\pi}{2}) = -sin(x) \text{ and } sin(x + \frac{\pi}{2}) = cos(x)$$

1) show that : $e^{-\frac{7\pi}{4}i} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

1) May we write : $i2 > i$?, or $i > 2i$?

1) Give the polar form of the numbers: $1 + i$, $-1 + i$, $1 - i$, $-1 - i$

1) show that he points: $1 + 2i$, $2 + 6i$, $2 + 11i$ are collinear.

1) calculate $(-1)^{-i}$.

1) can you explain : $2log(-1) = log(-1)^2 = log(1) = 0$?

1) give the value of $loglog(i)$.

1) show that the multiplication of z by i varies the argument by an angle $\frac{\pi}{2}$ by two methods.

1) show that $\bar{z} - z$ and $z - \bar{z}$ are pure imaginary.

1) show that : $i\begin{vmatrix} 1 & z_1 & \bar{z_1} \\ 1 & z_2 & \bar{z_2} \\ 1 & z_3 & \bar{z_3} \end{vmatrix}$ is a real number.

1) show by two methods that $\forall\theta \in \mathbb{R} : \left|e^{i\theta}\right| = 1$

1) show that : $arg(\theta_1\theta_2) = arg(\theta_1) + arg(\theta_2)$.

1) show that : $arg(\bar{z}) = -arg(z)$.

1) if $x \geq 0$, calculate $|x^z|$. What if $x < 0$?

1) prove : $|z| = |\bar{z}|$

1) show that for real α : $|\alpha \times z| = |\alpha| \times |z|$

1) show that $|zw| = |z||w|$ by two methods.

1) is it true that $|z^n| = |z|^n$ for $n > 2$?

1) prove : $z\bar{z} = |z|^2$, $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$

1) prove $\overline{z + w} = \bar{z} + \bar{w}$, $\overline{zw} = \bar{z}\bar{w}$

1) let α a real number is : $\overline{\alpha z} = \alpha.\bar{z}$ true ?

1) is the relation : $\overline{z^{-1}} = (\bar{z})^{-1}$ true ?

1) Show that : $\overline{(\frac{z}{w})} = \frac{\bar{z}}{\bar{w}}$.

1) show that: $\sqrt{a+bi} = \sigma + it \Longrightarrow \sigma^2 + t^2 = |a+ib|$, by two methods.

1) Calculate $\sqrt{1+i}$, then find $\tan(\frac{\pi}{8})$

1) when we have : $cos(\sigma) + isin(\sigma) < 0$?

1) when we have : $cos(t) + isin(t) = 0$?

1) when do we have : $cos(x) + isin(x) > 0$?

1) prove that : $cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$, $sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

1) prove : $(cos\theta + isin\theta)^n = \cos(n\theta) + isin(n\theta)$

1) find $cos\left(\frac{\pi}{12}\right)$ and $sin\left(\frac{\pi}{12}\right)$

1) $cos^2(t) = \frac{1+cos2t}{2}$ and $sin^2(t) = \frac{1-sin2t}{2}$

1) find then the antiderivatives of $cos^2(t)$ and $sin^2(t)$.

1) find $cos\left(\frac{\pi}{24}\right)$ and $sin\left(\frac{\pi}{24}\right)$

1) is $f(x) = \frac{|x|}{x}$ continuous ?

1) show that $f(z) = z^2$ is continuous for every z. Is $f(z) = \bar{z}$ continuous?

1) if $z = \sigma + it$, find dz .

1) show if f is derivable then $f'(z) = \frac{du}{d\sigma} + i\frac{dv}{d\sigma} = \frac{du}{dt} - i\frac{dv}{dt}$

1) show that $\lim_{z\to 0} \frac{\bar{z}}{z}$ does not exist

1) is $\bar{z}$ derivable at the point $z = 0$?

1) show by two methods that $e^{\bar{z}}$ is not differentiable.

1) find : $log'|z|$.

1) show that if u and v verify the **Cauchy-Riemann** equation, then they verify the **Laplace** equation.

1) what do we mean by : $|z| \to +\infty$?

1) write $f(z) = z^2$ in polar form . Find $(z^2)'$ by the limit method.

1) is $f(z) = z + i$ holomorphic ? and $g(z) = z + t$?

1) let $f(\sigma, t) = \frac{\sigma}{t} + \frac{t}{\sigma}$. Find f_σ and f_t.

1) is the function $\frac{\sigma - t}{\sigma}$ defined at $(0,0)$? continuous at $(0,0)$?

1) show that

$$f(\sigma, t) = \frac{t\sigma}{t^2 + \sigma^2}$$

is not continuous at the point $(0,0)$.

1) consider **Peano**'s example, show that $f_{t\sigma}$, $f_{\sigma t}$ are not continuous.

1) show that : $\frac{dz}{d\theta} = iz$.

2

ζ– Graphs

One cannot discard graphs at all in mathematics; they can give good information on how a function varies in its domain of definition and can lead to some ideas on the behavior of a function and how to deal with it when solving problems.

Real graphs

In real analysis we cannot present a function graphically if the number of variables is ≥ 3 . For example we can easily graph the function

$$z = f(x, y)$$

and get a surface, but we cannot graph the function

$$t = g(x, y, z)$$

Worse, we cannot draw the graph of a complex function $f(z)$ because it needs a space of 4 dimensions : 2 for the variable $\sigma + it$ and 2 for $u(\sigma, t) + iv(\sigma, t)$, in general this means the same as writing

$$\mathbb{C}^n = \mathbb{R}^{2n}$$

In our case : $\mathbb{C}^2 = \mathbb{R}^4$, but anyway we can visualize some of its features by:

a) Drawing the real function $M = |\zeta(z)|$ as a function of real variables σ and t

$$M: (\sigma, t) \longrightarrow |\zeta(z)|$$
$$\mathbb{R}^2 \longrightarrow \mathbb{R}$$

the graph will be in 3 dimensions space $\mathbb{R}^3$ like any real function of two variables $z = f(x, y)$ and get a surface:

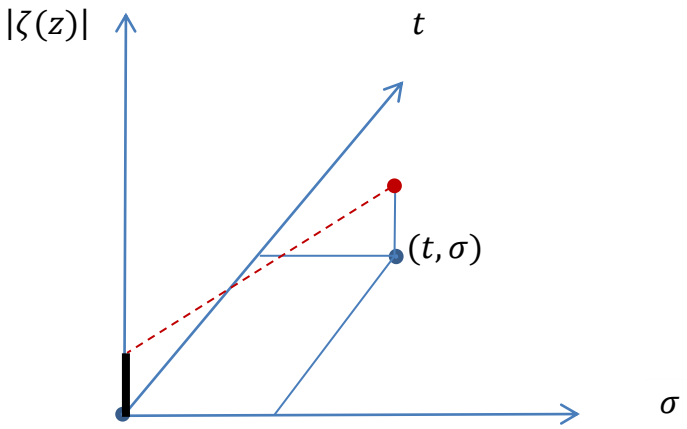


Fig. $|\zeta(z)| = f(\sigma, t)$

b) Drawing the real function $M = |\zeta(z)|$ as a function of u and v

$$\begin{array}{rcl} M: (u, v) & \longrightarrow & |\zeta(z)| \\ \mathbb{R}^2 & \longrightarrow & \mathbb{R} \end{array}$$

And get also a surface as in the figure:

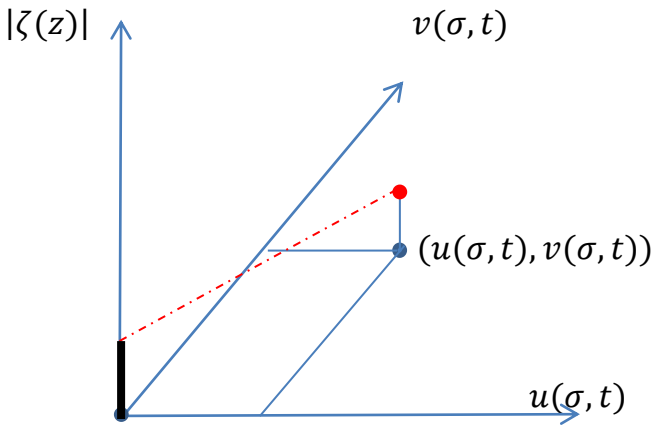


Fig. $|\zeta(z)| = f(u, v)$

It is not very different from the precedent, it is just a matter of scaling only.

c) By drawing the real function $v = f(u)$

$$\begin{array}{rcl} f: \quad u & \longrightarrow & v = f(u) \\ \mathbb{R} & \longrightarrow & \mathbb{R} \end{array}$$

d) by drawing the real function $u = f(v)$ as a transform of $\sigma = g(t)$ i.e. seeing it as an application that transform a domain in $\mathbb{R}^2$ to another domain in $\mathbb{R}^2$

$$M: (\sigma, t) \longrightarrow (u, v)$$
$$\mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

Thus we transform for example a curve $\sigma = g(t)$ to another curve $v = g(u)$, and these sort of transformations are very adequate if we precisely use complex numbers instead of real numbers. Since ζ is holomorphic, this transformation is a *conformal mapping*, because it preserves the angles between intersected curves.

e) In the case where the variable z is a pure real number$(t = 0)$

$$\zeta: \quad x \longrightarrow \sum_1 \frac{1}{n^x}$$
$$]1, +\infty[\longrightarrow]1, +\infty[$$

we have the following graph :

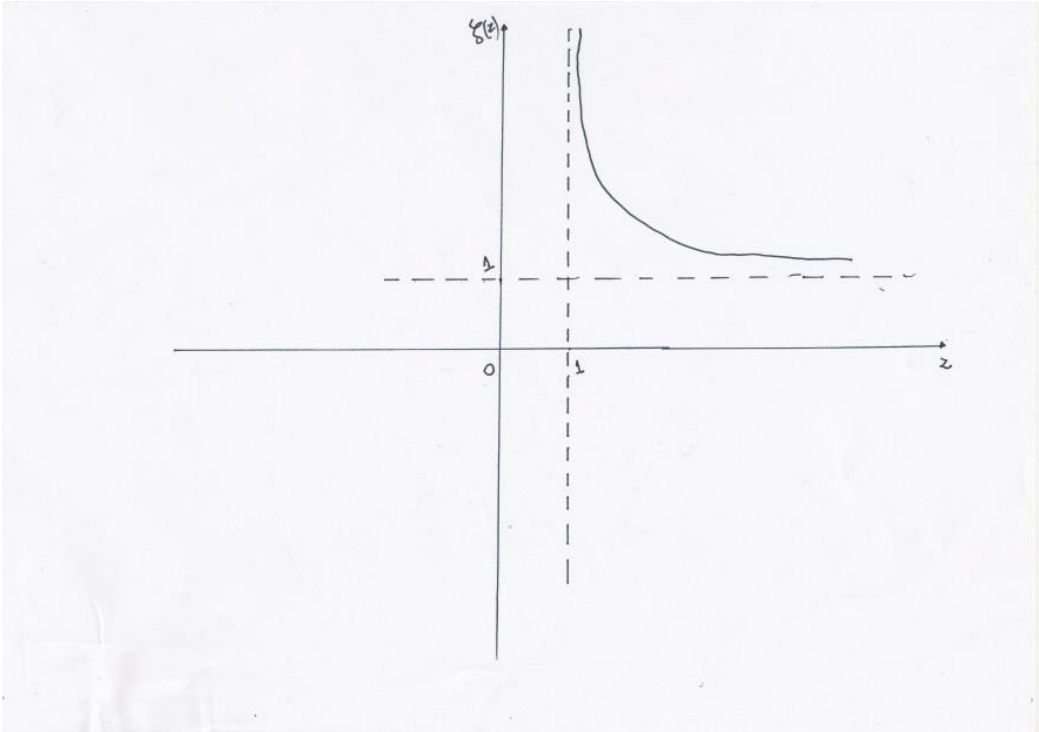


Fig. The graph of the function ζ for real z

Periodicity and symmetry

A function f is periodic of period r if

$$f(x+r) = f(x)$$

very known functions in mathematics to be periodic are the trigonometric functions : cos and sin , and thus any function presented by a **Fourier** series is periodic

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos(nx) + \sum_{n=1}^{\infty} b_n sin(nx)$$

simply because : for any given n and x

$$cos(nx\ ) = cos(nx\ + 2\pi k)$$

and

$$sin(nx\ ) = cos(nx\ + 2\pi k)$$

thus

$$cos\big(n(x\ + 2\pi k)\big) = \cos(nx + 2\pi N) = \cos(nx)$$

And

$$sin\big(n(x\ +2\pi k)\big) = sin(nx + 2\pi N) = sin(nx)$$

which mean that $f(x + 2\pi k) = f(x)$.

For the zeta function this is complicated because of two factors: the $log(n)$ and the presence of another variable σ in the denominator. Since

$$n^{-z} = n^{-\sigma - it} = e^{(-\sigma - it)\log n} = \frac{cos(\text{tlogn})}{n^{\sigma}} - i\frac{sin(\text{tlogn})}{n^{\sigma}}$$

that gives

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{cos(tlogn)}{n^{\sigma}} \quad - \quad i\sum_{n=1}^{\infty} \frac{sin(tlogn)}{n^{\sigma}}$$

so for the infinite series $\sum_{n=1}^{\infty} \frac{cos(tlogn)}{n^{\sigma}}$ for example, and even if $\sigma = \sigma_0$ we have

$$\frac{cos((t + 2n\pi)logn)}{n^{\sigma_0}} = \frac{cos(tlogn + log(n)2\pi k)}{n^{\sigma_0}} \neq \frac{cos(tlogn)}{n^{\sigma_0}}$$

So the function ζ is not periodic even for the same $\sigma = \sigma_0$. So we have simply :

$$\zeta(\sigma_0 + i(t + 2\pi k)) \neq \zeta(\sigma_0 + it)$$

The values taken by ζ are symmetric to the line $\Im(z) = 0$. Using a propriety of complex functions, known to **Euler**, namely that

$$f(\sigma - it) = u - iv$$

and known as:

Schwarz's principle

Let f a function of complex variable z then :

$$\overline{f(z)} = f(\bar{z})$$

as illustrated in the figure

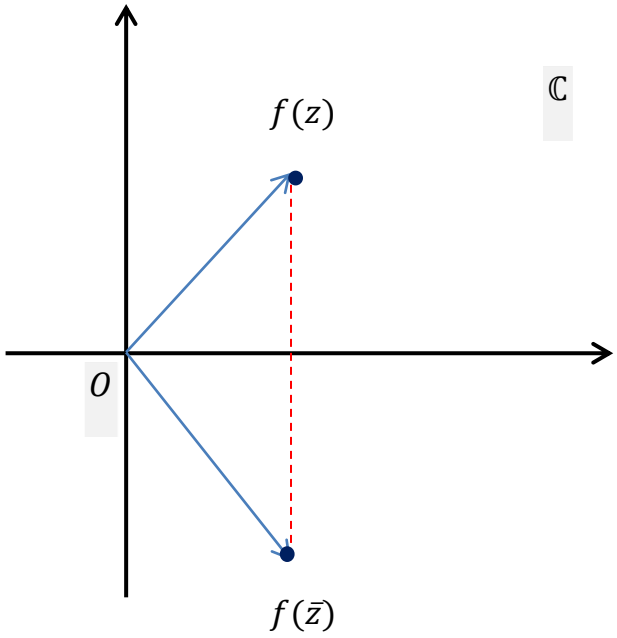


Fig. by conjugation we find: $\overline{f(z)} = f(\bar{z})$

so

$$\zeta(\sigma - it) = \sum_{n=1}^{\infty} \frac{cos(tlogn)}{n^{\sigma}} \quad + \quad i \sum_{n=1}^{\infty} \frac{sin(tlogn)}{n^{\sigma}}$$

Values taken by ζ

If f is real numerical function i.e. from $\mathbb{R}$ to $\mathbb{R}$, then we know that it may take as images either the whole $\mathbb{R}$ or only a part of it , and we have many examples like:

$-$ The trigonometric functions $\cos(x)$ and $\sin(x)$ take all the value from 1 to -1 infinitely many times.

$-$ the function $f(x) = 2x^2 - 1$ takes all the values from -1 to infinity two times.

$-$the function $g(x) = x + 1$ takes all the real values only once.



For functions of complex variable, this is not the case and we have two important theorems due to French mathematician Emile **Picard** $(1856 - 1941)$, one of them will be seen later for entire functions :

***Picard** little theorem(1879) :*

Every non-constant meromorphic function f can take all the complex values in $\mathbb{C}$ except perhaps two values.

as an example of this result is the meromorphic function

$$f(z) = tan(z) = \frac{sin(z)}{cos(z)}$$

which does not take the values $+i$ and $-i$.

Pure real and imaginary

We may also observe the possibility that ζ can be pure real or pure imaginary in the domain $\Re(z) > 1$. In the first case we look for values of σ and t such that

$$\sum_{n=1}^{\infty} \frac{sin(tlogn)}{n^{\sigma}} = 0$$

This is possible, for the of definition of our function ζ i.e. $\sigma > 1$:

$$\left|\sum_{n\geq 1} \frac{1}{n^{z}}\right| \leq \sum_{n\geq 1} \left|\frac{1}{n^{\sigma}}\right| = \sum_{n\geq 1} \frac{1}{n^{\sigma}}$$

and

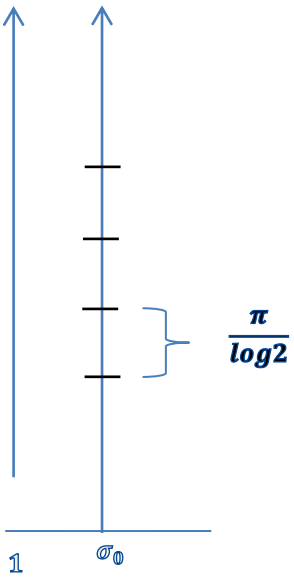
$$\lim_{\sigma\to+\infty} |\zeta(z)| = 1$$

So ζ cannot be complex or pure imaginary for some $\sigma_0 > \sigma > 1$, and the biggest term

$$\frac{sin(\text{tlog2})}{2^{\sigma}}$$

vanishes at $t = k\frac{\pi}{log2}$, which gives a difference between pure real lines by the quantity

$$\frac{\pi}{log2}$$



so $\zeta(z)$ cannot be pure imaginary in the domain $\Re(z) > 1$, i.e. there are no values of σ and t such that

$$\sum_{n=1}^{\infty} \frac{cos(tlogn)}{n^{\sigma}} = 0$$

Exercises

1) Is the following functions periodic

$$tan(z), \qquad cotan(z)\,, \qquad \frac{1}{cos(z)}, \qquad \frac{1}{sin(z)}$$

if yes, what are the periods?

1) find the expression of t_0 (i.e. vertical difference) where

$$cos(tlogn) = cos(tlogn + 2\pi k) \quad and \quad sin(tlogn) = sin(tlogn + 2\pi k)$$

1) can we have periodicity in the horizontal difference for a give T and every σ ? why?

1) show that the **Schwarz** principle is true for real functions.

1) any holomorphic function $f(z)$ is considered as a transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$, what sort of transformations when σ =constant are given by the functions :

$*)$ z

$*)$ z^2

$*)$ $\frac{1}{n^z}$

1) a complex number z is a *fixed point* if $f(z) = z$. Find the fixed point for the homographic function

$$f(z) = \frac{az + b}{cz + d}$$

1) find he values of : $e^{\pm i2n\pi}$, $e^{z\pi i}$, $e^{z\pi}$

1) for what values of z is the function $1 - 2^{1-z}$ real? imaginary?

1) solve the equations for σ and t :

$$\begin{cases} 1 - 2^{1-\sigma}\cos(tlog2) = 0 \\ 2^{1-\sigma}\sin(tlog2) = 0 \end{cases}$$

1) can they be pure real ? pure imaginary?

3

55

Complex infinite series

Now we analyze the function ζ considered as a convergent infinite series

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

and as a function of a complex variable z

$$\zeta(z) = \prod_{p} \left(1 - \frac{1}{p^z}\right)^{-1} = iv(\sigma, t) + iv(\sigma, t)$$

The function of complex variable s, that will be represented by these two expressions, when they converge, I designate by $\zeta(s)$. Both do not converge unless the real part is greater than 1.

Convergence:

For infinite series and products, the problem that we always face first is the problem of *convergence*. To prove that a complex *sequence*

$$c_n = a_n + ib_n$$

converges, we must show that both real part a_n and imaginary part $b_n$converge, for a complex *numerical* series : $\sum_{n\geq 0} c_n = \sum_{n\geq 0}(a_n + ib_n)$

Theorem

If a complex infinite series $\sum_n c_n$ converge then $\lim_{n\to+\infty} c_n = 0 + i0$

this means that both

$$a_n \longrightarrow 0 \quad \text{and} \quad b_n \longrightarrow 0$$

Then it is better to write

$\sum_n c_n$ converges then $lim_{n\to+\infty}|c_n| = 0$

and then construct the series The both terms go to 0 but the same for a complex *series*

$$\sum_{n\geq 0} c_n = \sum_{n\geq 0}(a_n + ib_n)$$

we take now an example

$$c_n = \frac{1}{n} + i\frac{2^n}{n!}$$

And form our infinite series

$$\sum_{n\geq 0} c_n = \sum_{n\geq 0}(\frac{1}{n} + i\frac{2^n}{n!})$$

which diverges.

so we a result of convergence for sequences:

Theorem:

If $|c_n|$ converges then c_n converges too. The converse is not true.

we take now an example

$$c_n = \frac{1}{e^{n+in}}$$

We see that

$$\sum_{n\geq 0} |c_n| = \sum_{n\geq 0} \frac{1}{e^n}$$

wich converges. for a function the variable z

$$\sum_{n\geq 0} f_n(z) = \sum_{n\geq 0} (u_n + iv_n)$$

to prove that it converges , we show that both real part and imaginary part

$$\sum_{n\geq 0} u_n \quad , \quad \sum_{n\geq 0} v_n$$

must converge. as

$$e^z = \sum_{n\geq 0} \frac{z^n}{n!}$$

In the real case, we first begin with the *necessary* condition of convergence:

Theorem:

If the infinite real series $\sum_n u_n$ converges then

$$\lim_{n\longrightarrow+\infty} u_n = 0$$

It is naturel to find out that the number

$$\sqrt{(\sum_{0}^{\infty} a_n)^2 + (\sum_{0}^{\infty} b_n)^2}$$

,which is the $|\sum_n c_n|$, is finite, a difficult task then we use other tricks.
As in the real case, also in the complex case there are series that converge but do not converge absolutely. We speak about *absolute value* real case, in the realm of complex variable z we say *module*.

Theorem:

If $\sum_n |c_n|$ converges then $\sum_n c_n$ converges too.

as

$$\sum_n \sqrt{{a_n}^2 + {b_n}^2} \geq \sum_n a_n$$

And

$$\sum_n \sqrt{{a_n}^2 + {b_n}^2} \geq \sum_n b_n$$

Then

$$\sum_n |c_n| \geq \sum_n c_n$$

this follows from the triangular inequality

$$\left|\sum_n c_n\right| \leq \sum_n |c_n|$$

Riemann says that the $\sum_{n=1}^{\infty} \frac{1}{n^z}$ and $\prod_p \left(1 - \frac{1}{p^z}\right)^{-1}$ both converge when $\Re(z) > 1$. Then it suffices that $|a_n|$, in our case $\left|\frac{1}{n^z}\right|$, converges. We have

$$\left|\frac{1}{n^z}\right| = |n^{-z}| = |e^{-zlogn}| = n^{-\sigma}|cos(tlogn) - isin(tlogn)| = n^{-\sigma}$$

according to the triangular inequality we have

$$\left|\sum_{n=1}^{\infty} \frac{1}{n^z}\right| \leq \sum_{n=1}^{\infty} \left|\frac{1}{n^z}\right| = \sum_{n=1}^{\infty} \frac{1}{n^\sigma}$$

The series of the form $\sum_{n=1}^{\infty} \frac{1}{n^\sigma}$ is called ***Riemann's*** *series* and $\sum_{n=1}^{\infty} \frac{1}{n}$ is the very known *harmonic series* , we know since **Oresme**(1320-1382) that it diverges.

There are by many methods to prove that, I have found a trick by which we can easily prove that the series $\sum_{n=1}^{k} \frac{1}{n}$ diverges by observing that

$$\sum_{n=1}^{\infty} \frac{1}{n} > \sum_{n=2}^{\infty} \int_n^{n+1} \frac{t^2 - 1}{t^2} dt > \sum_{n=2}^{\infty} \int_n^{n+1} \frac{1}{t} dt = +\infty$$

But it is useful a general and practical tool :

Theorem (integral test):

$\int_1^{+\infty} f(x)dx$ converges iff $\sum_{n=1}^{\infty} f(n)$ converges

so

$$\int_1^{+\infty} \frac{1}{x^\sigma} dx = \left. \frac{x^{1-\sigma}}{1-\sigma} \right|_1^{+\infty}$$

which converges for $\sigma > 1$ (when $x \longrightarrow 0$) and equals to $\frac{1}{1-\sigma}$.

The product $\prod_p (1 - \frac{1}{p^z})^{-1}$ converges too, because for $\sigma > 1$ we know that :

Theorem:

If $\sum_n |c_n|$ converges , the product $\prod_n (1 - c_n)$ converges.

Since when $\sigma > 1$ we have $\forall\, k$

$$\sum_{p=2}^{k} \frac{1}{p^\sigma} < \sum_{n=1}^{k} \frac{1}{n^\sigma}$$

and when $k \longrightarrow +\infty$

$$\sum_{p=2}^{\infty} \frac{1}{p^{\sigma}} < \sum_{n=1}^{\infty} \frac{1}{n^{\sigma}} < +\infty$$

then $\prod_p (1 - \frac{1}{p^z})^{-1} < +\infty$.

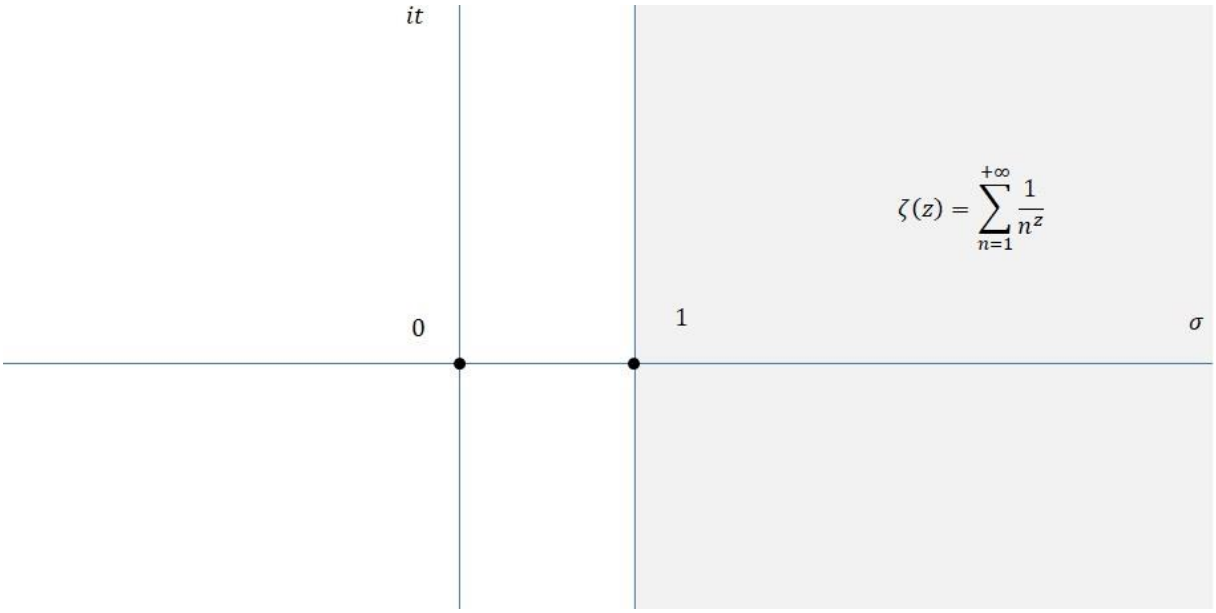


Fig. Domain of convergence of the zeta function.

Uniform convergence

Another very important propriety of the series $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$ is that it converges *uniformly* for $\Re(z) > 1$. Or the same : it converges on every finite compact. Uniform convergence permits us to do very useful operations later. But uniform converge is also a difficult propriety to prove on we use another stronger propriety

Definition:

We say the a function series $\sum_{n=1}^{\infty} f_n(x)$ converges *normally* if there exist a positive sequence a_n such that for all x, $|f_n(x)| \leq a_n$ and

$$\sum_{n=1}^{\infty} a_n < +\infty$$

We say also that $f_n(x)$ is *uniformly bounded*. Similarly, the complex series $\sum_{n=1}^{\infty} f_n(z)$ converges *normally* if there exist a positive sequence a_n such that for all x, $|f_n(z)| \leq a_n$ and

$$\sum_{n=1}^{\infty} a_n < +\infty$$

If the series converges *normally* then there exists a theorem due to **Weierstrass** named the $M - test$:

Theorem (**Weierstrass'** M test):

If there exists a real sequence a_n such that *for all z*: $|f_n(z)| \leq a_n$ and

$$\sum a_n < +\infty$$

alors $\sum f_n\,(z)$ converge uniformly.

so

$$\sum_{n=1}^{\infty} f_n(z) \text{ are uniformly bounded } \Longrightarrow \sum_{n=1}^{\infty} f_n(z) \text{ converges uniformly}$$

So for the series $\sum_{n=1}^{\infty}\left|\frac{1}{n^z}\right|$, it converges normally when $\sigma > 1$, for $\forall z$ such that $1 < \varepsilon < \sigma$ we have

$$\sum_{n=1}^{\infty}\left|\frac{1}{n^z}\right| \leq \sum_{n=1}^{\infty}\frac{1}{n^{\varepsilon}} < +\infty$$

And it converges uniformly for any z such that $\Re(z) > 1$.

Inversions

If the infinite series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly we have always for infinite series the following identity

$$\frac{d}{dx}\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty}\frac{df_n(x)}{dx}$$

and also

$$\sum_{n=1}^{\infty}(\int f_n(x)\,dx) = \int(\sum_{n=1}^{\infty} f_n(x))dx$$

that means we can ***invert*** the infinite sum $\sum$ with $\frac{d}{dx}$ and $\int$. The reader has to remark :

i) that uniform convergence is ***sufficient*** condition and ***not necessary*** for that.

ii) the resulting series and integral must also converge uniformly in order that the inversion is valid.

With this we can also now show that ζ is holomorphic for $\sigma > 1$ by using the **Cauchy-Riemann** equations to

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{\cos(\text{tlogn})}{n^{\sigma}} \quad - \quad i\sum_{n=1}^{\infty} \frac{sin(\text{tlogn})}{n^{\sigma}}$$

we find

$$\frac{\partial u}{\partial \sigma} = \sum_{n=1}^{\infty} \frac{cos(\text{tlogn})logn}{n^{\sigma}} = \frac{\partial v}{\partial t}$$

and

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \frac{-sin(\text{tlogn})logn}{n^{\sigma}} = -\frac{\partial v}{\partial \sigma}$$

Then again the function ζ is holomorphic for $\sigma > 1$

Exercises

1) Could you find another function f and a, b such that $\int_b^a f(x)dx = \frac{1}{n}$

1) show that for the real case : $\sum_n u_n$ converges then $\lim_{n\to+\infty} u_n = 0$

1) Prove that : $|zw| = |z||w|$.

1) show that $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$

1) prove that $|z + w| \leq |z| + |w|$ by two methods.

1) prove that $|a + ib| \leq |a| + |b|$.

1) prove that $|z_1 + \cdots + z_n| \leq |z_1| + \cdots + |z_n|$.

1) if $\sum_1^\infty |z_n| < +\infty$, prove that $|\sum_1^\infty z_n| < +\infty$.

1) prove that $|z - w| \leq |z| + |w|$.

1) prove that $|w| - |z| \leq |w - z|$.

1) prove that $|w| - |z| \leq |w + z|$.

1) show that : $|w| - |z| - |v| \leq |w + z + v|$.

1) show that for $|z| = R$ we have : $\frac{1}{R^2+R+1} \leq \frac{1}{|z^2+z+1|}$

1) show that: $\frac{1}{R^2+R+1} \leq \left|\frac{1}{z^2+z+1}\right| \leq \frac{1}{R^2-R-1}$

1) show that for $|z| = 3$ $\frac{11}{10} \leq \left|\frac{z^2+1}{z^2+2}\right| \leq \frac{7}{8}$

1) and for $|z| = 4$ show that $\frac{3}{5} \leq \left|\frac{z+i}{z-1}\right| \leq \frac{5}{3}$

1) show that $|e^z - 1| \leq e^{|z|} + 1$.

1) then prove that : $|e^z - 1| \leq \left|e^{|z|} - 1\right| \leq |z|e^{|z|}$. use:

1) prove that $|z| \leq |x| + |y| \leq \sqrt{2}|z|$.

1) prove that $|z + w| + |z - w| \leq 2|z|^2 + 2|w|^2$.

1) what does mean : $|z - z_0| < \varepsilon\ ,\ \varepsilon > 0$? draw it.

1) what does mean : $0 < |z - z_0| < \varepsilon\ ,\ \varepsilon > 0$? draw it.

1) draw the set of point z such that $|z - 1| < 3$, where the circle intersects Ot?

1) draw the set of point z such that $1 \leq |z + 4| < 2$

1) draw the set of point z such that $3 \leq |z - 2| < 4$, where the circle intersects Ot?

1) solve $|z + 1| = 1$

1) how many solution does the equation $|z| = 1$ have ?

1) show that : $x \prod_n (1 - \frac{x}{n^2})$ converges for all real x

1) show that for every prime $p \in \mathbb{P}$

$$(1 - \frac{1}{p^z})^{-1} = 1 + \frac{1}{p^z} + \frac{1}{p^{2z}} + \cdots$$

1) show that for every prime $p \in \mathbb{P}$

$$\sum_1 \frac{1}{n^z} = \prod_p (1 + \frac{1}{p^z} + \frac{1}{p^{2z}} + \cdots)$$

1) now show that for every prime $p \in \mathbb{P}$

$$\prod_p \left(1 + \frac{1}{p^z} + \frac{1}{p^{2z}} + \cdots\right) = \prod_p (1 - \frac{1}{p^z})^{-1}$$

1) thus, show that for every prime $p \in \mathbb{P}$

$$\sum_1 \frac{1}{n^z} = \prod_p (1 - \frac{1}{p^z})^{-1}$$

1) where are the series $g(z) = \sum_1 n^{-1-z}$ is defined ?

1) the series $\sum_0 \frac{1}{e^{n+in}}$ and show both its real and imignary series converge.

1) show that the $\sum_1 (-1)^n log n$ diverges

1) for what value the $\sum_1 \frac{(-1)^n log n}{n^z}$ converges ?

1) for what value the $\sum_1 \frac{1}{n^{1-z}}$ converges ?

1) prove by another method that $\sum_{n\geq 1}^{\infty}\frac{1}{n}$ diverges

1) calculate $\sum_{0}^{1000} i^n$.

1) calculate $\sum_{0}^{+\infty} i^n$.

1) calculate $\sum_{0}^{n}\cos(k\theta)$ and $\sum_{0}^{n}\sin(k\theta)$. Hint : use $\sum_{0}^{n} z$ with e^{in}.

1) find thus : $\sum_{0}^{n} a^k\cos(kx)$ and $\sum_{0}^{n} a^k\sin(kx)$.

1) calculate $\sum_{-\infty}^{+\infty}\frac{1}{n^2+1}$, $\sum_{-\infty}^{+\infty} n^3$, $\sum_{-\infty}^{+\infty}\frac{1}{n+i}$

1) does the series $\sum_{n\geq 0}^{\infty}\frac{1}{nlog(n)}$ converge?

1) prove that $log\prod_p (1-\frac{1}{p})^{-1}$ diverges.

1) Show that series : $\sum_{n=1}^{\infty}\frac{\cos(\text{tlogn})}{n^\sigma}$ and $\sum_{n=1}^{\infty}\frac{\sin(\text{tlogn})}{n^\sigma}$ converge uniformly for $\sigma > 1$.

1) does the series $\sum_{n=1}^{\infty}\frac{\log(n)}{n^\sigma}$ converges for $\sigma \geq 1$?

1) show that both

$$\sum_{n=1}^{\infty}\frac{sin(\text{tlogn})logn}{n^\sigma} \quad , \quad \sum_{n=1}^{\infty}\frac{cos(\text{tlogn})logn}{n^\sigma}$$

converge uniformly.

1) let $c_n = \frac{1}{n} + i\frac{1}{n^2}$, give the sums :

$$\sum_{n=1}^{\infty} c_n \quad , \quad \sum_{n=1}^{\infty}|c_n| \quad \text{and} \quad \left|\sum_{n=1}^{\infty} c_n\right|$$

if they exist. Give a relation between the three sums.

1) prove that if the complex $\sum_{n=1}^{\infty} c_n$ converges than : $\lim_n c_n = 0$

1) let a c_n complex sequence , $\lim_n|c_n| = 0$ prove that

$$\lim_n \Re(c_n) = 0 \quad \text{and} \quad \lim_n \Im(c_n) = 0$$

1) show that if $\sum_n c_n$ converge , then $|\sum_n c_n|$ converges.

1) let c_n be a complex sequence, prove that if $\sum_n |c_n|$ converges then $\sum_n c_n$ converges .

1) is there a complex sequence c_n , such that $|\sum_n c_n|$ converges and $\sum_n |c_n|$ does not converge?

1) is there a complex sequence c_n , such that $|\sum_n c_n|$ converges and $\sum_n c_n$ does not converge?

1) how can we compare between $\sum_{n=1}^{\infty} |n^{-z}|$ and

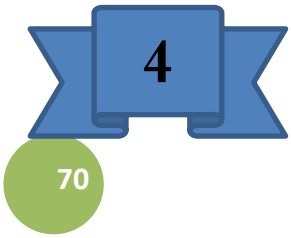
$$\sqrt{(\sum_{n=1}^{\infty} \Re(n^{-z}))^2 + (\sum_{n=1}^{\infty} \Im(n^{-z}))^2}$$

1) calculate : $\zeta'(z)$ for $\sigma > 1$.

1) calculate : $log\zeta(z)$ for $\sigma > 1$.

1) calculate : $\frac{\zeta'(z)}{\zeta(z)}$ for $\sigma > 1$.

1) how many solution does the equation $|z| = 1$ have ?

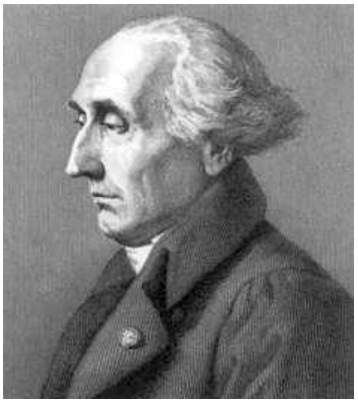


ζ as an analytic function

The importance of complex analysis resides in the *theory of analytic functions*. A great thing we know about the function ζ is that it is a *analytic* complex function in its domain of definition, i.e. where the infinite series $\sum n^{-z}$ converges.

Taylor series

Founded on a new unifying philosophy introduced in the XVIII century by Italian mathematician **Lagrange**(1736-1813), the theory of analytic functions began for the known operations in the real case



Founded on the good supposition that any real function can be written as a sum of an infinite series

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots$$

When x becomes $x + iy$, we will see new and stupendous phenomena. The known operations in the real case : infinites series, integration, derivation,… ..are treated by a very unified manner.

Worried by foundational issues of Calculus by giving a meaning to $f'(x)$, and through his experience with functions like

$$cos(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$
$$sin(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Lagrange found that polynomials are very *good* and manageable functions defined for all real numbers, they are the only functions that could be calculated by hand, that can be especially for all other function.

Precisely in his book *Fonctions Analytiques* (1797) , he defined a function to be *real analytic* if it can be represented by a ***Taylor*** *series*

$$f(x) = \sum_0^{+\infty} a_n(x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots$$

where $a_n = \frac{f^{(n)}(x_0)}{n!}$, which give us a definition for the derivative through :

$$f^{(n)}(x_0) = n!\, a_n$$

Taylor series were introduced previously by English mathematician Brook **Taylor**(1685-1731) in 1715 . A **Taylor** series , as we will see, is a special case of *power series*

$$\sum_{0}^{+\infty} a_n (x - x_0)^n$$

Lagrange also introduced his ***Lagrange*** *rest* in the approximation of a function f with a series of **Taylor**

$$f(x) = \sum_{0}^{n} a_k (x - x_0)^k + R_n(x)$$

Definition

If a function f can be represented by a series

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

that has a sense (i.e. converges) then f is *analytic* , and we have

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

A function is analytic at x_0 if it is analytic in all *neighborhood* of x_0 i.e. in any interval of the form $]x_0 - \varepsilon_1, x_0 + \varepsilon_2[$, and we say that the **Taylor** series is defined *around* a

given point x_0 . The expression $x - x_0$ means that in a *neighborhood* of x_0 , an interval like $]x_0 - \varepsilon_1, x_0 + \varepsilon_2[$ for $\varepsilon_1, \varepsilon_2 > 0$.

To make it easier, we usually chose an interval like

$$]x_0 - \varepsilon, x_0 + \varepsilon[$$

a function is analytic over $[x_0, x_0 + \varepsilon[$ only is not analytic at x_0.

If $x_0 = 0$ we get the more simple series , ***Maclaurin*** *series(1748)*

$$f(x) = \sum_0^{+\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

all polynomials of finite degree are analytic functions.

Examples : here are some known real analytic functions with their domains of convergence

i) $cos(x) = \sum_{n\geq 0}^{\infty} (-1)^n \frac{x^{2n}}{2n!}$, for all $x \in]-\infty, +\infty[$

ii) $sin(x) = \sum_{n\geq 0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$, for all $x \in]-\infty, +\infty[$

iii) $e^x = \sum_{n\geq 0}^{\infty} \frac{x^n}{n!}$, for all $x \in]-\infty, +\infty[$

vi) $\frac{1}{1-x} = \sum_{n\geq 0}^{\infty} x^n$, for $(|x| < 1)$

v) $log(1+x) = \sum_{n\geq 1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$, for $|x| < 1$

Power series

From **Taylor** series , in case a_n are well defined , we move to power series, the

generalization of **Taylor** series, where a_n are arbitrary. These series can be of real variable x

$$\sum_{n\geq 0}^{\infty} a_n\,(x-x_0)^n$$

or more general of the complex variable z

$$\sum_{n\geq 0}^{\infty} a_n\,(z-z_0)^n$$

Power series are a very interesting special case of infinite function series, and unlike them , they have many interesting proprieties. The coefficients a_n are the most important part of a power series, they may be given arbitrarily, or in the case of **Taylor** calculated by the formula $a_n = \frac{f^{(n)}(x_0)}{n!}$.

One of the essential proprieties of power series is that : a power series always converges uniformly in its domain of convergence; we have the famous **Abel**'s result:

Abel Lemma (18)

if $M > 0$ and $r > 0$ such that $|a_n|r^n < M$ then

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n$$

converges uniformly for $|z-z_0| < r$

Another useful propriety of power series is that the derivate of a power series is a power series and thus converge *uniformly* too

Theorem (18)

if $f(z)$ is a power series then

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$$

moreover f' converges uniformly in D and R is the same.

This comes easily from **Abel**'s lemma, by differentiating term by term a uniformly convergent series, which shows again that an analytic is derivable infinitely many times. In the real case, the inverse is not true , not every C^{∞} function is analytic as an example if we consider the function

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is continuous over $\mathbb{R}$ and derivable infinitely many times, but is not analytic over $\mathbb{R}$ because we have $\forall n$:

$$f^{(n)}(0) = 0$$

and

$$\sum_{n=0}^{\infty} a_n (x - 0)^n = 0$$

for all x which is not true for any $x \neq 0$ if $f(x) = e^{-\frac{1}{x^2}}$. we remark her although is

C^{∞}, it does not assure that it is analytic, whereas in the case of complex functions it suffices that $f'(z)$ exists.

It comes also that

Theorem:

If f is analytic on a domain D then

$$\int f(z)dz$$

is analytic over it .

as for a real variable x , also for complex number z , we look often if we can present a complex function by a power series , which is more interesting :

i) $cos(z) = \sum_{n\geq 0}^{\infty} \frac{z^{2n}}{2n!}$

ii) $sin(z) = \sum_{n\geq 0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$

iii) $e^{z} = \sum_{n\geq 0}^{\infty} \frac{z^{n}}{n!}$

iv) $\frac{1}{1-z} = \sum_{n\geq 0}^{\infty} z^{n} \ \ (|z| < 1)$

v) $log(1+z) = \sum_{n\geq 1}^{\infty} \frac{(-1)^{n-1}}{n} z^{n}$, $|z| < 1$,...



are analytic functions in their domain of definition D.

The domain of convergence of a real power series can be either an interval or a point. its radius of convergence R may be calculated by **Cauchy-Hadamard** formula :

Theorem

Let a_n be the coefficients of a power series , then the radius of convergence R is given by the formula:

$$R = \overline{\lim_{n\to\infty}} \frac{1}{\sqrt[n]{|a_n|}}$$

Or by using older **d'Alembert** criteria

$$R = \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = b$$

if $b < 1$ the series converges

if $b > 1$ the series diverges

if $b = 1$ we cannot decide

The real and complex infinite series and their domain of convergence:

a) In the real case, an infinite series has for domain of convergence interval

$$\sum_1 \frac{1}{n^x}$$

or two disjoint intervals

$$\sum_1 \frac{1}{n^{|x|}}$$

b) In the real case , an infinite power series has an interval

$$\sum_0 x^n$$

or the whole plan

$$\sum_0 \frac{1}{n!} x^n$$

c) the complex case it has either a disk like

$$\sum_0 z^n$$

or the whole plane like

$$\sum_n \frac{1}{n!} z^n$$

The point of view of **Lagrange** has been developed further by **Weierstrass** :

Theorem:

A function f of a complex variable z is analytic in a domain D iff it is developable by a power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

that converges in its domain D.

a product and the sum of two or more analytic function is an analytic function. We now have this great result

Theorem

If f is a complex function over a domain D then :

$$f \text{ is holomorphic on } D \iff f \text{ is analytic on } D$$

which means that : if $f'(z)$ exists in a domain D then $f(z)$ is developable in a power series in D.

From this we can get another corollary

Theorem

If f is a complex analytic function on D and

$$f(z = \sigma + it) = u(\sigma, t) + iv(\sigma, t)$$

then both u and v are real analytic functions.

The function ζ is not a power series by definition, but it can be developed in such a series according to the previous theorem as:

$$\zeta(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

The idea of **Weierstrass** is to work with disk of convergence of a convergent power series and theoretically an interesting result of continuation: a continuation by analycity:

Theorem of analytic continuation:

Let f_1 and f_2 two analytic functions on D_1 and D_2 respectively , then :

if

$$f_1 = f_2 \quad \text{over} \quad D_1 \cap D_2$$

then

$$f_1 = f_2 \quad \text{over} \quad D_1 \cup D_2.$$

What is very important is the continuation is unique : if f_1 and f_2 are two continuations of $g(z)$, then we have surely $f_1(z) = f_2(z)$. Another fact from this as we shall see, is if f is analytic then its zeros(roots) are isolated.

The eta function

As an example of analytic continuation, we took the eta -function or **Direchlet** eta function , which is defined for $\Re(z) > 0$ by the infinite series

$$\eta(z) = 1 - \frac{1}{2^z} + \frac{1}{3^z} - \frac{1}{4^z} + \cdots$$

We have

$$\frac{1}{n^z} - \frac{1}{(n+1)^z} = z \int_n^{n+1} \frac{1}{x^{z+1}} dx$$

But, for a given $\in \mathbb{C}$:

$$\left|\int_n^{n+1} \frac{1}{x^{z+1}}dx\right| \leq \int_n^{n+1} \frac{1}{|x^{z+1}|}dx = \int_n^{n+1} \frac{1}{x^{\sigma+1}}dx < \frac{1}{n^{\sigma+1}}$$

The series $\sum_1 \frac{1}{n^{\sigma+1}}$ converges for $\sigma > 0$ and so does $\sum_1 \frac{(-1)^{n-1}}{n^{\sigma}}$.

From the function η we can get the function ζ, for we have for $\Re(z) > 1$ with absolute convergence

$$\eta(z) = 1 + \frac{1}{3^z} + \frac{1}{5^z} + \frac{1}{7^z} + \cdots - \frac{1}{2^z} \times (1 + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \cdots)$$

and since $\zeta(z) = \sum_0 \frac{1}{(2n+1)^z} + \sum_1 \frac{1}{(2n)^z}$, then

$$\eta(z) = -\frac{1}{2^z} \times \zeta(z) + \zeta(z) - \frac{1}{2^z} \times \zeta(z)$$

which gives

$$\eta(z) = (1 - 2^{1-z})\zeta(z)$$

thus we have partially continuated ζ , namely only to the domain $\mathcal{R}(z) > 0$, by using **Dirichlet** function, the pole of is canceled by the zero of the factor $(1 - 2^{1-z})$. so:

Theorem

The function η is defined for $\mathcal{R}(z) > 0$ by

$$\eta(z) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n^z} = (1 - 2^{1-z})\zeta(z)$$

is analytic .

but η is still defined pour $\Re(z) > 0$ by the theorem of analytic continuation gives

$$\zeta_2(z) = \frac{\eta(z)}{(1 - 2^{1-z})}$$

Then $\zeta_1(z)$ and $\frac{\eta(z)}{(1-2^{1-z})}$ are equal in the domain $\Re(z) = \sigma > 1$ then they are the same in the domain $\Re(z) > 0$. And over the domain $\sigma > 1$

$$\zeta_1(z) = \zeta_2(z)$$

The function η has zeros, they are the zeros of the factor $(1 - 2^{1-z})$ and there are an infinite number of them. Although the two series

$$\eta = \sum_{n\geq 1} \frac{sin(tlogn)}{n^\sigma}(-1)^{n-1} + i\sum_{n\geq 1} \frac{sin(tlogn)}{n^\sigma}(-1)^{n-1}$$

are not periodic at the line $\sigma = \sigma_0$ as we have seen with ζ, curiously the function η has periodic zeros on the line $\sigma = 1$.

Singularities:

A function f may be not analytic at a point z where it is not defined, i.e. becomes infinite, or has other anomalies. These all points are called *singularities*

If f is not defined at a finite set of singularities $z_1, \dots, z_n$, then the power series is replaced by a new series named ***Laurent*** *series ,* attributed to the French mathematician **Laurent**(1813-1854) it is seen as a generalization of **Taylor** series

Theorem of **Laurent** (1843), **Weierstrass**(1841):

If a complex function f has a singularity at $z = z_0$ then

$$f(z) = \dots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 \dots$$

$$= \sum_{-\infty}^{+\infty} a_n (z-z_0)^n$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz$$

Any **Laurent** series is composed of two parts :

i) The infinite series $\sum_{-\infty}^{-1} a_n (z-z_0)^n$ which is called *the principal part* , *that may have only finite number of terms.*

ii) and $\sum_{0}^{+\infty} a_n (z-z_0)^n$ a complex **Taylor** series.

Singularities of a complex function may be classified in two big classes : *isolated* singularities and *non-isolated* singularities.

Isolated singularities

a) removable singularity

it may be eliminated et on a

$\lim_{z \to z_0} f(z)$ exists and finite.

like the function

$$f(z) = \frac{\sin(z)}{z}$$

A
t the point $z = 0$ with

$$\lim_{z \to 0} \frac{\sin(z)}{z} = 1$$

It may be also called *fake* or *false singularity*.

c) Poles:

The term "*pole*" is due to **Briot** and **Bouquet** . If the principal part has only a finite number of terms(or only finite non zero terms) the singularity is a pole and the function f is called *meromorphic* (from $mero$: 'part') it has only a finite number of poles. The pole may of order n if the principal part is of the form

$$\dots + 0 + 0 + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)}$$

and has the residue a_{-1}. If only then it is a simple pole, the function

$$f(z) = \frac{1}{z-1}$$

Has a simple pole $z = 1$ for it can be written

$$f(z) = \cdots + 0 + \frac{0}{(z-1)^2} + \frac{1}{z-1} + a_0 + a_1(z-1) + \cdots$$

Some functions can have infinity many singularities in every disk , like

$$f(z)=\sum_{0}^{\infty}\frac{1}{z^n-1}$$

d) Essential singularity

The term "*essential*" is due to **Weierstrass**. If the number of terms in the principal part is infinite, the singularity z_0 is called *essential singularity* which may be seen as a pole of infinite order since it has infinity many terms $\frac{a_n}{(z-1)^n}$, then it cannot be eliminated. Essential singularities were introduced by the French mathematician Victor **Puiseux** (1820-1883) in 1850. As an example the function

$$f(z)=e^{\frac{1}{z}}$$

has for essential singularity the number $z=0$.

Non Isolated singularities

these are like the *branch point* that occurs in the multivalued functions

$$log(z)\ ,\ \ z^{\frac{1}{n}}$$

A theorem due to **Picard** we have seen already says that any function with essential singularity may take any value in $\mathbb{C}$. This is the **Picard** great theorem , we will see another theorem about entire functions subsequently. **Riemann** himself as one of the founders of the complex analysis, his philosophy was to define complex functions by their singularities.

The power series for ζ

the function ζ is analytic over $\mathbb{C}$ with a simple pole at the point $z = 1$, so the expression for this function for $\sigma > 1$

$$\zeta(z) = \frac{1}{(z-1)} + a_0 + a_1(z-1) + \cdots$$

The pole is $a_{-1} = 1$.

Theorem

For all $z \neq 1$ we have

$$\zeta(z) = \frac{1}{z-1} + \sum_{0}^{+\infty} \frac{\gamma_n}{n!}(z-1)^n$$

Where the numbers γ_n, are the ***Stieltjes*** *constants*

$$\gamma_n = \lim_{l \to +\infty} \left(\sum_{k=1}^{l} \frac{(\log k)^n}{k} - \frac{(\log l)^{n+1}}{n+1} \right)$$

So

$$a_n = \frac{\gamma_n}{n!}$$

Exercises

1) evaluate the sum of $\sum_{n\geq 0}^{\infty} e^{-n}$.

1) show that if $f(z)$ is analytic in D then it is continuous in D.

1) show that every polynomial $p(x)$ of degree n is analytic over $\mathbb{R}$.

1) indicate the singularities of the real sequences

$$f_n(x) = \frac{1}{nx} \ , \quad g_n(x) = \frac{1}{n-x} \ , \quad h_n(x) = \frac{1}{\log(nx)}$$

1) suppose that a $f(x)$ is developed by **Maclaurin** series, show that $a_n = \frac{f^{(n)}(0)}{n!}$

1) give the power series for $f(x) = \sqrt{1+x}$ by two methods , and give the domain of convergence

1) show that

$$f(x) = \begin{cases} e^{-x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is infinitely continuously differentiable in $\mathbb{R}$. We say that f is $C^{\infty}(\mathbb{R})$. Why it is not analytic?

1) let

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

show that $f^{(n)}(0) = 0$, show thus that f is not analytic over $\mathbb{R}$, but it is analytic over $\mathbb{R}^*$.

1) what about

$$f(x)=\begin{cases} x^2\sin(\frac{1}{x}) & x\neq 0 \\ 0 & x=0 \end{cases}$$

1) show that if $f(z)$ is analytic then $-f(z)$ is analytic too.
1) is any analytic function holomorphic?
1) show that the function $\eta(z)$ is analytic by two methods.
1) show that $\Re\eta$ and $\Im\eta$ are harmonic.
1) find the zeros of the function $\eta(z)$.
1) may the series $1-\frac{1}{2^\sigma}+\frac{1}{3^\sigma}-\frac{1}{4^\sigma}+\cdots$ vanishes for

$$0<\sigma<1 \quad ?$$

1) what is $\lim_{z\to 1}\zeta(z)\,(1-2^{1-z})$
1) calculate $\eta(1)$.
1) find $\eta'(z)$.
1) show how the function η diverges for $|z|\longrightarrow 0$
1) do : $\sum_1 \frac{(-1)^{n+1}}{n}(-1)^{n-1}$ and $\sum_1 \frac{(-1)^{n}}{n}(-1)^{n-1}$converge?
1) show that the series $\sum_{n=1}\frac{\cos(tlogn)}{n^\sigma}(-1)^{n-1}$ converges for $\sigma>0$.
1) show that $1-2^{1-\sigma}$ increases when $\sigma\in]0,1[$.
1) show that $1-2^{1-\sigma}<0$ for $\sigma\in]0,1[$.
1) what it is the sign of η ? thus the sign of $\zeta(z)$?
1) find the solutions of $1-2^{1-z}=0$. They are the roots of η.
1) are the spacing between η zeros the same? What is the period?
1) we want to show that : $\eta(1)=log(2)$. So find series for $log(1-x)$. Think of geometric series of x^n and the indefinite integration $\int\frac{1}{1-x}dx$.
1) find series for $log(1+x)$ and thus $\eta(1)$.
1) find now the sum : $\sum_1\frac{x^{-2n}}{2n}$
1) indicate the singularities of the sequences

$$f_n(z)=\frac{1}{(z+1)^n}, \quad g_n(z)=\frac{1}{1-z^n}, \quad h_n(z)=\frac{1}{1-|nz|}$$

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1) indicate the singularities of the functions

$$f(z)=\frac{1}{(z+i)^2+8}, \quad g(z)=\frac{1}{z^2+2} \ , \quad h(z)=\frac{z^2+2}{3z^2-2}$$

1) show that for $\sigma>1$: $\frac{2^z}{2^z-1}\sum_0 \frac{1}{(2n+1)^z}=\sum_1 \frac{1}{(2n)^z}$

1) prove that: $\sum_1 x^n=\frac{1}{1-x}$ for $|x|<1$

1) what is the power series of $log(1+x)$?

1) give the power series for : $\frac{1}{1+z}$

1) give the power series of : $\frac{1}{(1+z)^2}$, $\frac{1}{(1-z)^2}$

1) does $\sum_0 z^{-n}$ converges ?

1) show that if $\sum_0 a_n z^n$ converge, then a_n is bounded.

1) show that if $\sum_0 a_n z^n$ converge for z_1 then it converges for z_2 : $|z_2|\leq|z_1|$.

1) what is the radius converges $\sum_0 nz^n$?

1) show that $\sum_0 n!\, z^n$ converges only for 0.

1) give a power series that converges only for -1, only for i.

1) by **Cauchy**-**Hadamard** lemma test the convergence of the power series

$$\sum_1 \left(\frac{z}{n}\right)^n \qquad \sum_0 z^n \qquad \sum_0^\infty (nz)^n \qquad \sum_1 \frac{z^n}{n} \qquad \sum_1 \frac{z^n}{n^2}$$

$$\sum_0 \frac{z^n}{n!} \qquad \sum_0^\infty n(z)^n$$

1) find the singularities and the of convergence radius for the series

$\sum_1 \frac{z^n}{n^2}$

$\sum_1 \frac{(-1)^{n-1}}{|z|}$

$\sum_1 \frac{z^n}{n^2(1-z^n)}$

$\sum_1 \frac{(-1)^{n-1}}{(n-|z|)}$

$\sum_1 \frac{z^n}{n}$

$\sum_1 \frac{(-1)^{n-1}}{n|z|}$

$\sum_1 \frac{(-1)^n}{|z|^2}$

1) find the convergence radius R of the series $\sum_1 (3+(-1)^n)^n z^n$

1) determinate the **Laurent** series for

i) $f(z) = \frac{1}{z+2}$

ii) $(z-1)cos(\frac{1}{z+2})$

1) prove that the functions are analytic

i) $f(z) = constante$

ii) $f(z) = 1 - 2^{1-z}$

1) show that for $0 < \sigma \leq 1$

$$\eta(z) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} n^z$$

1) show that $\eta(z) < 0$ for $0 < \sigma \leq 1$

1) show that $\sum_{n\geq 1} \frac{(-1)^{n-1}}{n^\varepsilon} \neq 0, \ \forall \varepsilon \in \mathbb{R}$

1) show that the function

$$f(z) = e^{\frac{1}{z}}$$

has at $z = 0$ an essential singularity

ζ and number theory

The zeta paradise:

The importance of the function ζ is not only seen as a phenomenal and difficult function of complex variable. Its relation to prime numbers is very apparent, through the preceded formula

$$\frac{1}{\zeta(z)} = \prod_{p\geq 2}\left(p^{z} - 1 \right) p^{-z}$$

but it has turned to be very tight to number theory in general. Mathematicians' manipulations with this amazing function only in $\Re(z) > 1$ has deterred more than what they expected. We know today that the function carries a lot of information on number theory functions called *"arithmetic functions*" .and the relation cane be tow write new function in the form

$$g(z) = \sum_{1}^{+\infty} f(n) \times n^{-z}$$

As the first example is the ζ derivative, so for $\Re(z) > 1$ we have

$$\zeta'(z) = \sum_{1}^{+\infty} n^{-z} \times log(n)$$

which we note $n^{-z} \times lo\, g(n) = f_n(z)$.

The Euler totient function

Let $\Re(z) > 1$, then we have

$$\frac{\zeta(z-1)}{\zeta(z)} = \sum_{1}^{+\infty} n^{-z} \times \varphi(n)$$

Where $\varphi(n)$ is a function goes back to **Euler** , it give the number of coprime integers to n and less than it, it has the value

$$\varphi(n) = \prod_{i=1}^{k} (1 - \frac{1}{p_i})$$

Divisors of a number n

we have for $\Re(z) > 1$, the function

$$\zeta^2(z) = \sum_{1}^{+\infty} d(n) \times n^{-z}$$

The function $d(n)$ is the number of divisors of n . Very amazingly

$$\zeta^k(z) = \sum_1^{+\infty} d_k(n) \times n^{-z}$$

Where $d_k(n)$ is the number of ways in which n can be factorized . And we have for $\Re(z) > 1$

$$\frac{\zeta^3(z)}{\zeta(2z)} = \sum_1^{+\infty} d(n^2) \times n^{-z}$$

and last we have this time for $\Re(z) > 2$

$$\frac{1 - 2^{1-z}}{1 - 2^{-z}}(z-1) = \sum_1^{+\infty} a(n) \times n^{-z}$$

With $a(n)$ is the greatest odd divisor of n.

The number of different prime factors

For $\Re(z) > 1$

$$\frac{\zeta^2(z)}{\zeta(2z)} = \sum_1^{+\infty} 2^{\nu(n)} \times n^{-z}$$

Where $\nu(n)$ is the number of different prime factors in n.

The number of squared factors

Let $\Re(z) > 1$, then we have

$$\zeta(z) = \sum_{1}^{+\infty})n^{-z}$$

The Mobius function

Let $\Re(z) > 1$, then we have

$$\frac{1}{\zeta(z)} = \frac{1}{\sum_1^{+\infty} n^{-z}} = \prod_{2} (1 - p^{-z})$$

but

$$\prod_{2} (1 - p^{-z}) = 1 - \frac{1}{2^z} + \frac{1}{3^z} - \frac{1}{5^z} + \frac{1}{6^z} - \frac{1}{7^z} + \frac{1}{10^z} - \cdots$$

We remark that the $p^k, k > 1$ are missing, then we define a new function

$$\begin{cases} 0 \; if \; p^k | n \\ 1 \quad if \; n = product \; of \; even \; number \; of \; different \; primes \\ -1 \quad if \; n = product \; of \; odd \; number \; of \; different \; primes \end{cases}$$

Let $\Re(z) > 1$, then we have

$$\frac{1}{\zeta(z)} = \sum_{1}^{+\infty} \frac{\mu(n)}{n^z}$$

The function $\mu(n)$ is called the ***Mobius*** *function*, we see it later because **Riemann** will use a form of it. We have also $\Re(z) > 1$

$$\frac{\zeta(z)}{\zeta(2z)} = \sum_{1}^{+\infty} |\mu(n)| n^{-z}$$

We will also that $\mu(n)$ has something to do with the **Riemann** hypothesis.

The von Mangoldt function

From $\zeta(z) = \prod_2 (1 - p^{-z})$ let

$$log\zeta(z) = -\sum_{p} log(1 - \frac{1}{p^z})$$

Knowing that $log(1-x) = \sum_1 \frac{1}{nx^n}(-1)^{n-1}$ then

$$log\zeta(z) = \sum_{n=1} \sum_{p=2} \frac{1}{np^{nz}}$$

But if we integrating $log\zeta(z)$ we find then

$$\frac{\zeta'(z)}{\zeta(z)} = -\sum_{n=1}\sum_{p=2}\frac{\log(p)}{p^{nz}} = -\frac{\log(2)}{2^z} - \frac{\log(3)}{3^z} - \frac{\log(5)}{5^z} - \dots - \frac{\log(2)}{2^{2z}} - \frac{\log(3)}{3^{2z}} - \frac{log(5)}{5^{2z}} - \dots$$

And we want to write it in the form $\sum_{n=1}\frac{f(n)}{n^z}$, we remark that all products of different primes $p_1 \times p_2 \times \dots \times p_k$ are missing, only product of the same primes ${p_i}^r$. German **von Manglodt** defined a new function

$$\Lambda(n) = \begin{cases} logp \ if \ n = p^k \\ 0 \qquad if \ n = product \ \ of \ different \ primes \end{cases}$$

That gives us

$$\frac{\zeta'(z)}{\zeta(z)} = -\sum_{1}^{+\infty}\Lambda(n)n^{-z}$$

The function $\Lambda(n)$ is the ***Mangoldt*** *function*. We can also going back to $log\zeta(z)$ by integrating to find

$$log\zeta(z) = \sum_{1}^{+\infty}\Lambda_1(n)n^{-z}$$

where $\Lambda_1(n) = \frac{\Lambda(n)}{\log(n)}$[6] . We will also learn that **Tchybechev** defined

$$\psi(x) = \sum_{n<x}\Lambda(n)$$

[6] see **Titchmarsh**: p.4.

So we can write then

$$\zeta'(z) = \sum_{1}^{+\infty} f_n(z)$$

and

$$log\zeta(z) = \sum_{1}^{+\infty} s_n \times f_n(z)$$

where $s_n = \frac{\Lambda_1(n)}{\log(n)}$, and $f_n(z) = \frac{log(n)}{n^z}$. Also

$$\frac{\zeta'(z)}{\zeta(z)} = -\sum_{1}^{+\infty} t_n \times f_n(z)$$

where $t_n = \Lambda_1(n)$, and.

The Liouville function

Let $\Re(z) > 1$. We have then

$$\frac{\zeta(2z)}{\zeta(z)} = \sum_{1}^{+\infty} n^{-z} \times \lambda(n)$$

The function $\lambda(n)$ is the ***Liouville*** *function* defined as $\lambda(n) = (-1)^r$, $r = \omega(n)$ being the number of prime factors.

The Bernoulli numbers

Let then we have for $\Re(z) > 1$ we have

$$\zeta(n) = \frac{(2\pi)^{2n}(-1)^{n+1}B_{2n}}{2(2n!)}$$

and for $\Re(z) < 1$ we have

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}$$

We will see that has a relation in integral with the function $\frac{e^x}{e^x-1}$ and this function are analytic when $x \neq 0$ and written as

$$\frac{e^x}{e^x - 1} = \sum_{1}^{+\infty} \frac{B_n}{n!} x^n$$

Exercises

1) using the formula given above, give the values:

$$\varphi(0), \quad \varphi(1), \quad \varphi(5), \quad \varphi(20), \quad \varphi(49)$$

1) calculate the function $\lambda(n)$ for the previous values of φ. (we will see the function $\lambda(n)$ again later)

1) show that : $\zeta(z) = \exp(\sum_n \sum_p \frac{1}{np^{zn}})$. Hint: use the **Taylor** series of $log(x)$ around 1 then make suitable changes.

1) show that : $-z\zeta(1-z) = B_z$

1) give the values:

$$\mu(0), \quad \mu(1), \quad \mu(25), \quad \mu(20), \quad \mu(77)$$

1) why is :

$$\sum_{p}\sum_{n}\frac{1}{np^{zn}}=\sum_{n}\sum_{p}\frac{1}{np^{zn}} \qquad ?$$

1) integrate the infinite series $\frac{\zeta'(z)}{\zeta(z)}$ to get the new relation $\Lambda_1(n)=\frac{\Lambda(n)}{\log(n)}$.

1) calculate : $\zeta'(z)$ for $\sigma>1$. Explain in the real case the sign $(-)$.

1) show that :

$$\left[\frac{1}{\zeta(z)}\right]'=-\sum_{1}^{+\infty}\Lambda(n)n^{-z}\times\sum_{1}^{+\infty}\mu(n)n^{-z}$$

First extraction

However , it is easy to find for the function an expression that always remains valid.

Riemann is searching for a function that "always remains valid" , that is a function definite for all variable z , i.e. in all $\mathbb{C}$, and at the same time equals to ζ in the half plan $\mathcal{R}e(z) > 1$. It is just a *continuation* of the function ζ to $\mathcal{R}e(z) < 1$, this is called today by the name *analytic continuation* , as we have seen , we continued the function ζ to the half plane $\mathcal{R}e(z) > 0$ by using the function η

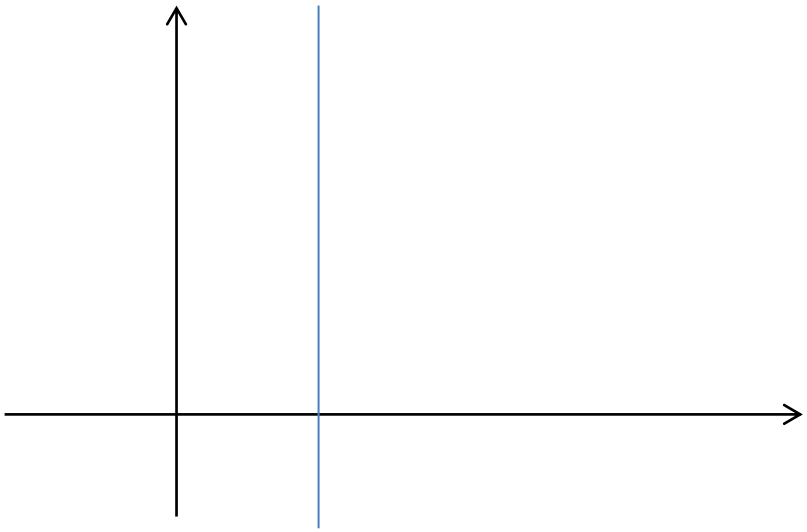


Fig. the real function η

It was not new at the time, because **Euler** already wrote la function gamma Γ originally defined for $x > -1$ only as :

$$\Gamma(x) = \int_0^{+\infty} e^{-t} t^x dt$$

It is seen as $I(x)$ a parameter x

$$\Gamma(x) = I(x) = \int_a^b F(t,x)dt$$

Can be written as the limit of an infinite product

$$\Gamma(x) = \lim_{n \to +\infty} \frac{n!}{x(x+1)\dots(x+n)} x^n$$

also

$$\Gamma(x) = \prod_n (1 + \frac{n}{x})^n$$

which are manifestly defined for all x , except when $x = -n$. This gives a *continuation* of Γ to the whole real line except for the points $x = 0, -1, -2, \ldots$

A classical simple example of this continuation is the function defined by the series

$$f(x) = 1 + x + x^2 + \cdots$$

the previous *geometric series* $\sum_{n\geq 0}^{\infty} x^n$ converges for $x < |1|$, but if we write

$$f(x) = 1 + x(1 + x + x^2 + \cdots) = 1 + xf(x)$$

we find the same

$$f(x) = \frac{1}{1 - x}$$

which is defined over $\mathbb{R}/\{-1\}$.

He will find a known procedure to extend the function ζ not only to the domain $\mathcal{R}(z) > 0$, but to the whole complex plane.

By use of the equation

$$\int_0^{\infty} e^{-nx}x^{s-1}\,dx = \frac{\Pi(s-1)}{n^s}$$

one gets first

$$\Pi(s-1)\zeta(s) = \int_0^{\infty} \frac{x^{s-1}}{e^x-1}dx$$

The gamma function due to **Euler** is a definite integral (improper integral) for real variable $t > 0$

$$\Gamma(t) = \int_0^{+\infty} e^{-x}x^{t-1}dx$$

Gauss was the first to write for complex variable $t = z$, and wrote for complex number z

$$\Gamma(z) = \Pi(z-1)$$

We have for $\Re(z) > 0$

$$\Gamma(z) = \int_0^{+\infty} e^{-x}x^{z-1}dx$$

It seems that **Riemann** thought like this : since the function Γ is defined for all complex numbers and since ζ is linked to it by the function

$$\Gamma(z)\zeta(z) = \int_0^{\infty} \frac{x^{z-1}}{e^x-1}dx$$

then it is possible that ζ itself may be continuated by the same way.

Riemann wrote $\Pi(z-1)$ (a notation due to **Gauss**) for our modern $\Gamma(z)$ due to **Legendre** (1798).
By the variable changing $x = nt$:

$$\Gamma(z) = \int_0^{+\infty} e^{-nt}(nt)^{z-1}dnt = n^z \int_0^{+\infty} e^{-nt}t^{z-1}dt$$

for n^z does not depend of t, that gives

$$\frac{\Gamma(z)}{n^z} = \int_0^{+\infty} e^{-nt}t^{z-1}dt$$

This equality is already known before **Riemann**, for example to **Euler** (1781). His teacher **Dirichlet**(1837) wrote

$$\frac{\Gamma(t)}{n^t} = \int_0^1 x^{n-1}\log^{t-1}(\frac{1}{x})dx$$

which, with a suitable variable change gives

$$\frac{\Gamma(t)}{n^t} = \int_0^{+\infty} x^{t-1}\mathrm{e}^{-nx}\,dx$$

Also his friend **Dedekind**(1852) wrote

$$\int_0^{+\infty} x^{\mu-1}\mathrm{e}^{-rx}\,dx = \frac{\Gamma(\mu)}{n^{\mu}}$$

Schlömlich (1858) found, for $0 < x < 1$, the expression

$$\frac{\Gamma(x)}{n^x}\sin(\frac{x\pi}{2}) = \int_0^{+\infty} t^{x-1}\sin(\mathrm{nt})\,dt.$$

Now, taking the infinite sum of the two members in the equality, **Riemann** found

$$\Gamma(\mathrm{z})\sum_{n=1}^{\infty}\frac{1}{n^z} = \sum_{n=1}^{\infty}\int_0^{+\infty} e^{-nx}x^{z-1}dx$$

$$= \int_0^{+\infty} e^{-x}x^{z-1}dx + \int_0^{+\infty} e^{-2x}x^{z-1}dx + \cdots$$

since the series $\sum_{n=1}^{\infty} e^{-nx}$ converges uniformly for $x > 1$, according to the $M - test$: $\forall\, x > 0$

$$\sum_{n=1}^{\infty}|e^{-nx}| \leq \sum_{n=1}^{\infty} e^{-n} < +\infty$$

we make enter the sum $\sum_{n=1}^{\infty}$ inside the integral (we say that we *interchange* the two signs $\sum$ and $\int$) , which gives for $\Re(z) > 1$

$$\Gamma(\mathrm{z})\sum_{n=1}^{\infty} n^{-z} = \int_0^{\infty}\sum_{n=1}^{\infty}(e^{-nx})\,x^{z-1}dx$$

but

$$\sum_{n=1}^{\infty} e^{-nx} = \frac{1}{e^x - 1}$$

the sum of a geometric series of ratio e^{-x}, which gives

$$\Gamma(z)\zeta(z) = \int_0^{\infty} \frac{x^{z-1}}{e^x - 1} dx\,, \qquad \Re(z) > 1$$

it appears that similar starting summation formulas have been existed in the work of **Tchebychev** in 1848 , i.e.



$$\int_0^{\infty} \frac{x^a}{(e^x - 1)e^x} dx = \Gamma(a+1) \sum_{n=2}^{\infty} \frac{1}{n^{a+1}}$$

who used it in his proof of the conjecture of **Bertrand**. (*Œuvres de Tchebchev* in French, printed in 18, *Tome Premier* : $p.2$) .

And also by **Abel** (see *Œuvres d'Abel* , in French (1839 and 1881) , printed in Christiania(*Oslo*), *Tome Premier* : $p.22$) who wrote

$$\int_0^{\infty} \frac{t^{2n-1}}{e^t - 1} dt = \Gamma(2n)(1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \cdots)$$

which is much more similar to **Riemann**'s , cf. **Edwards** [$p.$ 9]. **Abel** was searching for an expression for the ***Bernoulli*** *numbers* by using definite integrals (See **Gram**(1903)).

The definite or improper integral

$$\int_0^{\infty} \frac{x^{z-1}}{e^x - 1} dx$$

is convergent for $\Re(z) > 1$ like this:

$$\frac{x^{z-1}}{e^x - 1} \sim \frac{x^{z-1}}{e^x}$$

and

$$\int_0^{\infty} \frac{x}{e^x} dx < +\infty.$$

and we know that

$$\int_0^{\infty} \frac{1}{xe^x} dx = +\infty$$

For $x \sim xe^x$ near 0, which means that in order to have our integral converges : we must have $z - 1 > 0$ i.e. $z > 1$.

Exercises

1) show that the following series converge uniformly:

i) $\sum_{n=1}^{\infty} e^{-2nx}$

ii) $\sum_{n=1}^{\infty} e^{-nx}$

i) $\sum_{n=1}^{\infty} x^{n}$

ii) $\sum_{n=1}^{\infty} e^{-nx}$

1) how did we change the variable to get the **Dirichlet** equality?

1) by the method of chapter 1 , find by two methods

$$\int sin^2 dx \, , \int cos^2 dx \, , \int cos^3 dx$$

1) find the limits

$$\lim_{x \to 0} \frac{x}{e^x - 1}$$

And for $t \in \mathbb{R}$

$$\lim_{x \to 0} \frac{x^{t-1}}{e^x - 1}$$

1) find

$$\left| \frac{x^{z-1}}{e^x - 1} \right|$$

1) show that the integral $\int_0^{\infty} \frac{x^{z-1}}{e^x - 1} dx$ converges for $\Re(z) > 1$

1) use **Schlömlich** formula to find

$$\int_0^{+\infty} \sin(\mathrm{nt})\, dt = \frac{1}{n}$$

1) calculate

$$\int_0^{\infty} \frac{x^{-1}}{e^x - 1} dx$$

1) find:

i) $\int_{-1}^{1} xe^x dx$

ii) $\int_{1}^{+\infty} xe^{-x} dx$

iii) $\int_{-1}^{1} xe^x dx$

iv) $\int_{1}^{+\infty} x^3 e^{-x} dx$

1) show that

$$\int_0^{\infty} \frac{x}{e^x} dx < +\infty.$$

1) show that

$$\int_0^{\infty} \frac{1}{xe^x} dx = +\infty$$

1) find the sum of each infinite series

i) $\sum_{n=1}^{\infty} e^{-2nx}$

ii) $\sum_{n=1}^{\infty} e^{-\frac{n}{k}x}$, $k \neq 0$

iii) $\sum_{n=0}^{\infty} e^{-nx}$

7

Integration in the complex plan

If now we consider the integral

$$\int \frac{(-x)^{s-1}}{e^x - 1} dx$$

taken in the positive sense from $+\infty$ to $+\infty$, and around a domain of values that contains in its interior the value 0, but that does not contain any value of discontinuity of the function under the integration sign …

What **Riemann** will do here is an integration *in the complex plan*, by integrating the function $\frac{(-x)^{z-1}}{e^x-1}$, or by notation

$$\int_{+\infty-\varepsilon i}^{+\infty+\varepsilon i} \frac{(-x)^{z-1}}{e^x - 1} dx$$

through the path containing the set number from $+\infty + \varepsilon i$ to $+\infty - \varepsilon i$, where $x = \sigma + it$ is a complex number, and $\varepsilon > 0$ as in the figure

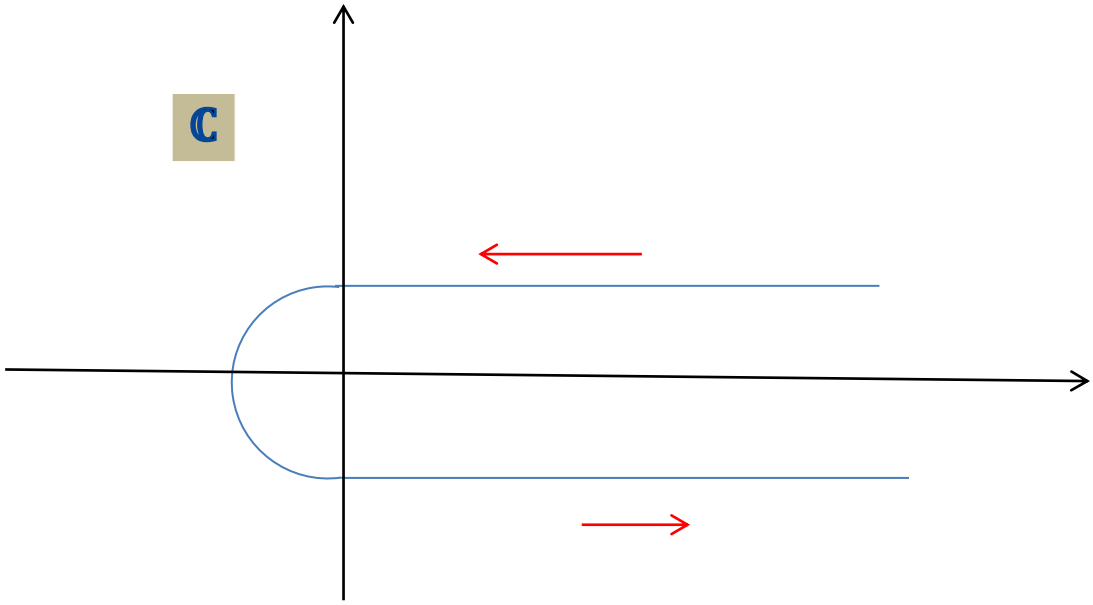


Fig. **Riemann**'s path of integration

The denominator $e^x - 1$ has infinitely many singularities of the form $\pm 2n\pi i$. Although **Poisson** was the first mathematician to carry an integration in the complex plan (1815 and 1820), the early elements of theory of *complex defined integral* , or integration in the complex plan has been introduced by **Cauchy** since 1814 , then published en 1827. Given two points z_1 and z_2 in the complex plan, the integral of a defined function f from z_1 to z_2 as in the real case, is the limit of infinite sum by the notation

$$\int_{z_1}^{z_2} f(z)dz = \lim_{\infty} \sum_{0}^{n} f(z_n)(z_{n+1} - z_n)$$

under some conditions that we shall see.

Calculation of definite integrals

Such integration in $\mathbb{C}$, does exit also in the real case and known as *line integration* : an integration of a function $f(x,y)$ through a path C situated in the real plan $\mathbb{R}^2$. the function f is seen as a *vectorial function*

$$f(x,y) \coloneqq P(x,y)\vec{\imath} + Q(x,y)\vec{\jmath}$$

and we ,in fact, integrate a *differential form*

$$\int_C P(x,y)dy + Q(x,y)dx$$

Where $P(x,y)$ and $Q(x,y)$ are real functions of x and y.
This situation comes naturally from, for example, the evaluation the *work of a force* $f\ (x,y)$ in the plan. Suppose that a particle that moves from a point (x_1, y_1) to another point (x_2, y_2). To calculate it , one can think of a parameter t between x and y and calculate a simple integral

$$\int_{t_1}^{t_2} g(t)dt$$

or

$$f(x,y) \coloneqq f(h(t),l(t))$$

Example:

We evaluate the line integral on the plan from the points $(x_1 = 0, y_1 = 0)$ and $(x_2 = 1, y_2 = 1)$, i.e. through a segment,

$$\int_{(0,0)}^{(1,1)} xdy + ydx$$

We like this: $x = t$ and $y = t$ then we only t. for the coordinates () and we ()=() then are .so the integral is

$$\int_{1}^{2} tdt =$$

The question is :

Question:

Does the value of the integral change if we take another different path between the two points?

It is known since **Clairaut**(1740) , **Euler**(1740) and **d'Alembert**(1768) that this integration does not depend on the path if the

$$P(x,y)dy + Q(x,y)dx$$

is exact, which means

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}$$

And a practical tool to calculate the integral is :

Green-Riemann theorem

$$\int_C Pdy + Qdx = \iint_D \left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\right) dxdy$$

Stated by the British mathematician and physicist **Green** (1793-1841) and proved by **Riemann** in 1851. In the plan $\mathbb{C}$, the operation is slightly different because the of the number i. But the domain D where we do integration is the same and have to some proprieties that today we name *topological*[7]:

i) The domain D is *connected*
ii) the domain D is *simply connected*, means that C does not cut itself
iii) D must be *regular*: that is if parallel or parallel to Ox we only in two points ..
iv) D must be connexe without holes.

When integrating in the complex plan , we ovoid singularities ,a fact that push us to *deform* our path of integration.

Line integration in $\mathbb{C}$ (which is in reality in the plan $\mathbb{R}^2$) is more free than integration on the real line $\mathbb{R}$. At the contrary of the real case, where there is only one path, there exists infinitely many *paths* between two complex points a and , and it is not very different because the function $f(z)$ is seen also as a vectorial function

[7] From Greek *topos* "place": the science of the how the points (elements) of a set are situated, the relation to each to the other, topology is from the point et of real numbers $\mathbb{R}$.

$$\int_C udx - vdy + i\int_C vdx + udy$$

The difference is that in complex variable, we can have the formula

$$\int_{z_1}^{z_2} f'(z)dz = f(z_2) - f(z_1)$$

which cannot obtained in the real case.

Paths and loops

To a simple real definite integral, To well characterize a path in $\mathbb{C}$, any C has two

i) C is open , it began from a point a and to a point b

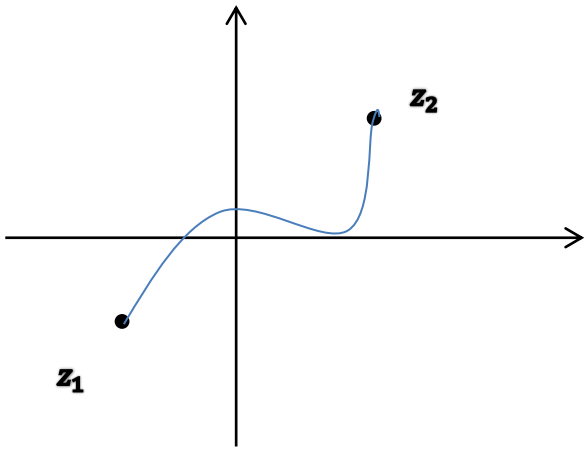


Fig. open path.

ii) C is close , then the problem is to know if there are singularities inside it

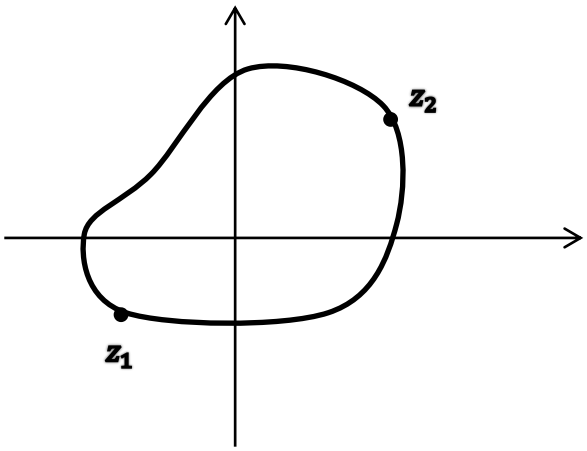


Fig. closed path.

we to substitute the numbers from z_1 to z_2 by a real parameter t and get a function $\gamma(t)$ of real variable . which gives

$$z = \gamma(t) = x(t) + iy(t)$$

That $f(z)$ became $f((\gamma\ (\mathrm{t}))$ and $dz = \gamma'(t)dt$ from which the integral

$$\int_C\ f(z)\,dz = \int f(\gamma\ (\mathrm{t}))\gamma'(t)dt$$

In the case when C is a closed path , t is named a *loop*, we can by the notation

$$\oint f(z)dz$$

The circle $\odot$ in the 'integral symbol means that integration is done in a closed contour. For a closed path (a loop) usually we use the equation of a circle or its halve, given that its equation is easy.

Observation

A circle is always presented by the simple transformation

$$z = \gamma\,(t) = e^{it} \quad \text{and} \quad dz = ie^{it}dt = izdt$$

for $0 \leq t \leq 2\pi$

If

$$\mathrm{M} = \max_C |f(z)|$$

then

$$\left|\int_C f(z)dz\right| \leq \int_C |f(z)|dz \leq M.l_C$$

where l_C is the length of the path C. Integration in the complex plan has a conventional *direction* and

$$\int_{C^+} \quad and \quad \int_{C^-}$$

are different :

- the *positive sens* : the *opposite* du sens du mouvement clockwise.

- the *negative sense* : the movement sense watch or anticlockwise.

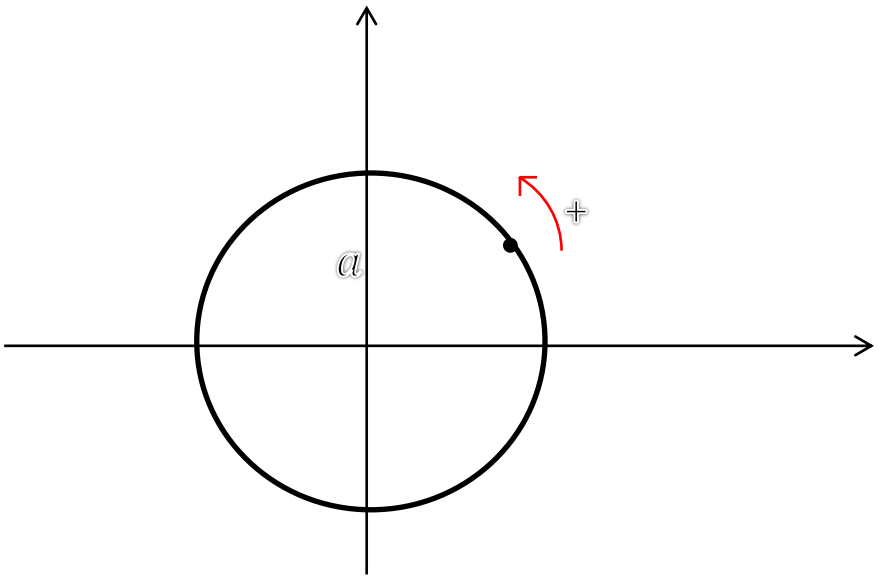
Besides in the real case too, it has a sense for

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

The convention comes from the circle in an equation of a circle of the radius a in a plane $\mathbb{R}^2$ or $\mathbb{C}$

$$y(t) = acos(t) + asin(t)$$

when the real variable t moves to in the usual positive sense from 0 to 2π the circle is drawn in the clockwise sense



In the complex plan to simplify we always work in a disc $D \subset \mathbb{C}$ of center a and of radius r .Also to find a good approximation. We will here give typical examples.

Example 1:

let's calculate the integrals:

1) $\int_{-1-i}^{1+i} z^2 dz = \frac{1}{3} z^3 \Big|_{-1-i}^{1+i} = \frac{1}{3} + i\frac{1}{3}$

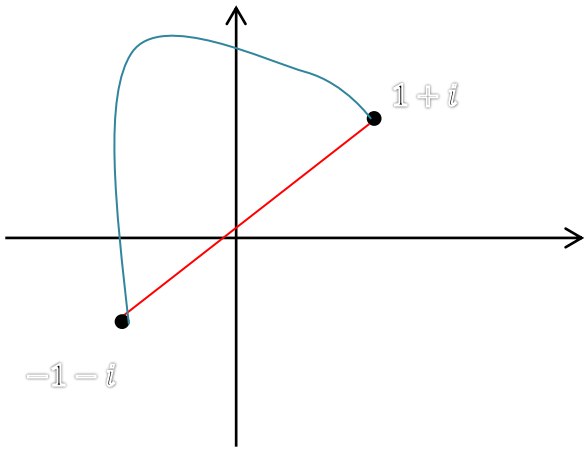


Fig. the integral $\int_{C_i} z\, dz$ does not depend on the path why?

always the same whatever the path is.

2) $\int_a^b z^n dz = \frac{1}{n+1} z^{n+1} \Big|_a^b$

3) $\int_C z^n dz = 0$ if C is closed, i.e. : $a = b$.

The path C is a path best presented by a circle, then the integral in the complex plan se is reduced to integration on the real line

$$\int_C f(z)\, dz = \int_0^{2\pi} e^{int}\, \mathrm{i}e^{it}\, dt = i\int_0^{2\pi} e^{it(n+1)}\, dt$$

$$= 0$$

Example2 :

The integral

$$\int_C \frac{1}{z} dz$$

Is a bit of problem because the domain D contains the value $z = 0$ for which $\frac{1}{z}$ is not defined. Then we ovoid this value and get

$$\int_C \frac{1}{z} dz = \int_0^{2\pi} \frac{1}{e^{it}} \, \mathrm{i}e^{it} \, dt = i2\pi$$

But if we get different values

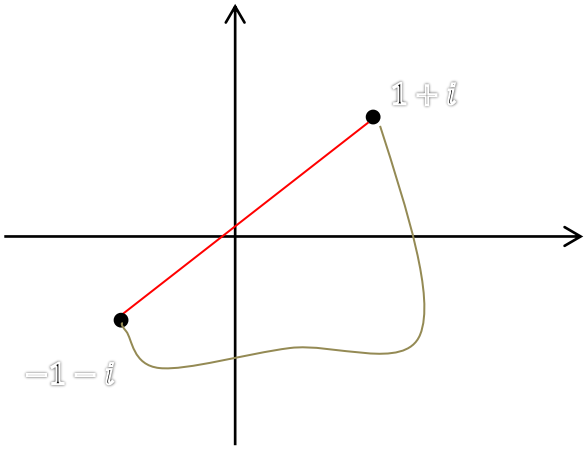


Fig. the integral $\int_{C_i} \frac{1}{z} dz$ is different according to the path why?

It has been proved that the value of the integral is the same between two points when the path C is *simple* (it does not intersect itself) :

Theorem

a function f holomorphic *inside* D , two paths C_1 and C_2 between two pints a, b then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

In a letter to **Bessel** in 1811, **Gauss** said

I affirm now that the integral $\int \phi(x)dx$ has only one value even if taken over different paths, provided that $\phi(x)$ is single-valued and does not become infinite in the space enclosed by the two paths.

For real vector fields this is equivalent to say: the line integral of an irrotational vector field around any closed path equals 0.

Exercises

1) show that

$$\int_C f(z)dz = \int_C udx - vdy + i\int_C vdx + udy$$

1) Why count we have a formula like $\int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1)$ in the case of real line integral.

1) show that the $\int_{+\infty-\varepsilon i}^{+\infty+\varepsilon i} \frac{(-x)^{t-1}}{e^x-1} dx$ converges for real t.

1) what about the value of $\int_{+\infty-\varepsilon i}^{+\infty+\varepsilon i} \frac{(-x)^{z-1}}{e^x-1} dx$ for complex numbers z.

1) say if the form are exact

a) $rdr - tdt$

b) $xdy + xdy$

1) find the $\int_0^1 \int_1^2 xydxdy$

1) find the value of

a) $\int_{(0,1)}^{(1,2)} xdx + ydy$

1) the use of complex number facilitate a lot of calculus machinery as an example the real integral

i) $\int_a^b e^{nx}\sin(nx)dx$ then compare with $\int_a^b e^{inx}dx$.

ii) $\int_a^b e^{nx}\cos(nx)dx$ then compare with $\int_a^b e^{inx}dx$.

1) give the values :

i) $\int_0^{\pi/2} \cos(z)\sin(z)dz$

ii) $\int_0^{+\infty} e^{-x^2}dx$

iii) $\int_{-\infty}^{+\infty} xe^{-x^2}dx$

iv) $\int_0^{+\infty} \frac{x}{e^x}dx$

v) $\int_{-\infty}^0 e^x dx$

vi) $\int_{-\infty}^0 e^{-x}dx$

vii) $\int_0^{+\infty} e^{-x}dx$

viii) $\int_0^{a\pm i\infty} e^{-z}dz$

ix) $\int_0^{\infty} e^{-z}dz$

x) $\int_0^{+\infty} e^{-z}dz$

xi) $\int_i^{1+i} \frac{1}{z}dx$

1) indicate the equivalent paths:

i) t , $t \in [0,1]$.

ii) $\sin(t)$, $t \in \left[0,\frac{\pi}{2}\right]$.

iii) $\cos(t)$, $t \in \left[0,\frac{\pi}{2}\right]$.

iv) $\sin(t)$, $t \in [0,\pi]$.

vi) $e^{2\pi it}$, $t \in \left[0,\frac{\pi}{2}\right]$.

1) integrate $f(z) = z$ on the curve $\gamma(t) = 1 + (i-1)t$ avec $0 \leq t \leq 1$ by two methods

1) integrate $f(z) = z^2 + 1$, on the unit circle by two methods.

1) calculate $f(z) = \int_1^{+\infty} \frac{1}{t^z} dt$, when does it converges?

1) what about $g(z) = \int_0^{+\infty} \frac{1}{t^{z-1}} dt$?

1) integrate $f(z) = logz$, on the unit circle by two methods.

1) calculate : $\int_1^{+\infty} z^n dt$.

1) then $\int_1^{+\infty} z^{-n} dt$.

1) show that : $\int_{a-i\infty}^{a+i\infty} \frac{dz}{z} = i\pi$. .

Cauchy theorem

We have seen of the real line integrations, we quote **Laugwitz** p. 74:

Questions related to the computation of real definite integrals contributed in a vital way to the rise of complex analysis. In the introduction to one of his papers of 1818, ***Cauchy*** *said that his investigations were an attempt to provide a rigorous justification of the methods of* ***Euler*** *(after 1759) and* ***Laplace*** *(after 1782) for the computation of real integrals using imaginaries.*

We provide

Riemann used complex analysis extensively in two works: in his on h theory complex (1851) and in his on **Abelian** integral(18)

Mondromy

Cauchy has obtained a lot of results in complex analysis, he early is defined monodrome[8] functions

Definition:

A function f on a complex is said to be *monodromic* I has one value in.

One a difference between a value of a function and path

One of the very surprising propriety of complex analysis is that if f is defined on a domain D , on a pour tout loop C in D, the following famous theorem

Theorem of **Cauchy** (18)

let f a holomorphic function in D and on C then:

$$\int_C f(z)dz = 0$$

We have seen already that f is holomorphic at a point z if f'exist at that point. This result is known under the name of *Cauchy Theorem*. And notice that it is conform with the Green-Riemann theorem for if the differential

$$Pdy + Qdx$$

is *exact* we have

$$\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} = 0$$

[8] See **Cauchy**, **Briot** , **Jordan.**

So

$$\int_C Pdy + Qdx = 0$$

or

$$\int_C f(z)dz = \int_C udx - vdy + i\int_C vdx + udy = 0$$

And we have seen before

$$\iint_D \left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\right) dxdy$$

and we have from the **Cauchy-Riemann** conditions

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y} \quad , \quad \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}$$

so

$$\int_C f(z)dz = 0$$

It was proved later by **Goursat** that is independent, and a corollary that if f as we have seen in chapter 1.

Goursat's lemma

If $f'(z)$ exits in D, then the integral does not depend of the path and we have

$$\int_C f(z)dz = 0$$

So the continuity of $f'(z)$ is not necessary.

But what if the domain is infinite?
The inverse of **Cauchy** theorem is a result due to the Italian mathematician Giacinto **Morera**(1856-1909).

Theorem of **Morera** (18)

let f a function that is *continuous* in a domain D, such that

$$\int_C f(z)dz = 0$$

then f is holomorphic on C.

another practical and important result in complex integration (known to **Gauss**) is the ***Cauchy*** *integration formula* (1825):

Theorem (1825)

Let f be a holomorphic function on a contour C then

$$f(w) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-w}\,dz$$

It means that when we know the values of a holomorphic function $f(z)$ on C we can know the value of $f(w)$ inside C, which is not true in the real case. So the term $z-a$ is to the plane where we always for convenience work in disks.

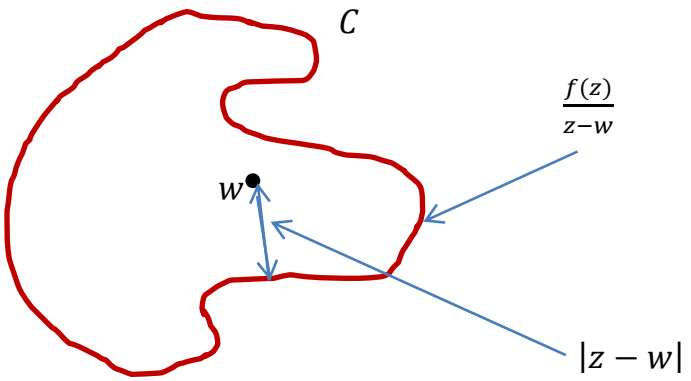


Fig. **Cauchy** integral formula

We see at one that it contains the *residue theorem*. It also implies the ***mean*** formula

Theorem (18)

let f be a holomorphic function the its value is:

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z + re^{ib})dt$$

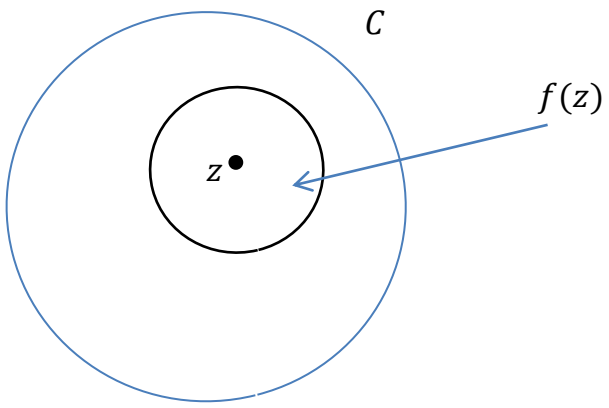


Fig. mean value of f around z.

This formula also implies again as we have mentioned before, that If f est derivable one then it is derivable a domain D infinitely many times in the same domain D , and its derivative is

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$$

The formula is useful to calculate integral thorough the practical formula

$$\int_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i}{n!}f^{(n)}(a)$$

these methods of integration in the complex plan serve even to evaluate some difficult real integrals.

Exercises

1) by the **Cauchy** integral formula evaluate the following integrals

$$\int \frac{e^z}{z-a}dz$$

$$\int \frac{\log(z)}{z-a}dz$$

$$\int \frac{z}{(z-a)^3}dz$$

1) show that $\int_C dz = l_C$

1) by integrating, find the formula

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int \frac{f(z)}{(z-a)^{n+1}}dx$$

1) show that $\int_C \frac{1}{(z-a)^n}dz = 0$ for $n \neq 1$

1) deduce that : $\left|f^{(n)}(a)\right| \leq M$

1) show that if $f(z)$ z complex, is analytic then it is holomorphic.

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The continuation of the function ζ

to the set $\mathbb{C}$

To continue the function to the whole $\mathbb{C}$, we follow the method used by the English mathematician **Titchmarsh** (1899-1963) (II, p.18) and without the **Riemann's** scary symbol $(-x)^{Z-1}$, so



$$\int_{C^+} \frac{w^{z-1}}{e^w - 1} dw$$

is a definite integral in the complex plan, here z and w are both

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complex numbers the integration in the domain, by C^+ we means taken in the positive direction i.e. "*counterclockwise*"

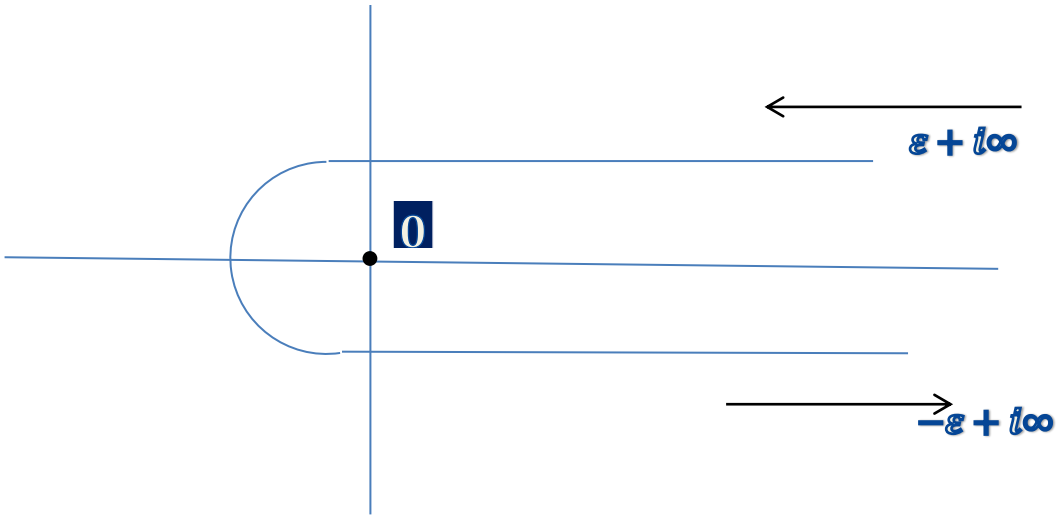


Fig. path integration through C^+

It is in function of , $I(w) = \int_{C+} \frac{w^{z-1}}{e^w - 1} dw$, that is similar to calculate an integral of on the real line

$$I(t) = \int_{\varepsilon}^{+\infty} \frac{x^{t-1}}{e^x - 1} dx$$

We have seen the function gamma $(t) = \int_0^{+\infty} e^x x^{t-1} dx$, the term z^{w-1} is as $e^{(w-1)logz}$. The logarithmic function is a *multiform* function and always poses the problem of the determination of its imaginary part :

Theorem

The logarithmic function $log(z)$ is a multiform function, and has infinity of imaginary parts such that

$$log(z) = log(|z|e^{i(\theta \pm 2k\pi)}) = log|z| + i(\theta \pm 2k\pi)$$

Already in the XVII century there was no agreement of the meaning of the logarithm of (negative and imaginary numbers), there was a controversy between **Leibnitz**(16-17) and jeans **Bernoulli**(16-17), of which **Euler** wrote two articles:

$\boldsymbol{a}$) *Sur les logarithms des nombres negatives et imaginaires* (1749).
b) *De la controverse entre m.* ***Leibnitz*** *and* ***Bernoulli*** *sur les logarithms des nombres*[9].

Bernoulli thought that

$$log(-x) = log(x)$$

Because $(-x)^2 = x^2$ implies $2log(-x) = 2\log(x)$. **Euler** pointed out that this means that

$$\left(x\sqrt{-1}\right)^4 = x^4$$

which implies that $log\sqrt{-1} = 0$, a contrary to a correct equality due to **Bernoulli** himself namely that $\frac{log(\sqrt{-1})}{\sqrt{-1}} = \frac{\pi}{2}$.

Leibnitz argumented by saying that if $log(-x) = log(x)$ then

$$0 = \log(-1) = -1 - \frac{1}{2} - \frac{1}{3} - \cdots$$

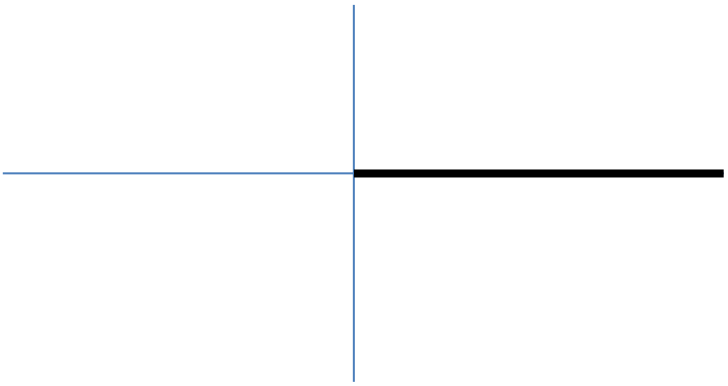
[9] See also ***Gregory*** *Work* (1865) p.124.

So **Euler** thought that $log(z)$ has infinitely many values, and in modern terminology then , we say that the function $\log(z)$ has many *branches.* (in fact, takes an infinity of real numbers).

Then to determine one principal must assignate a well determined value to the argument varies in

$$0 < \theta \leq 2\pi$$

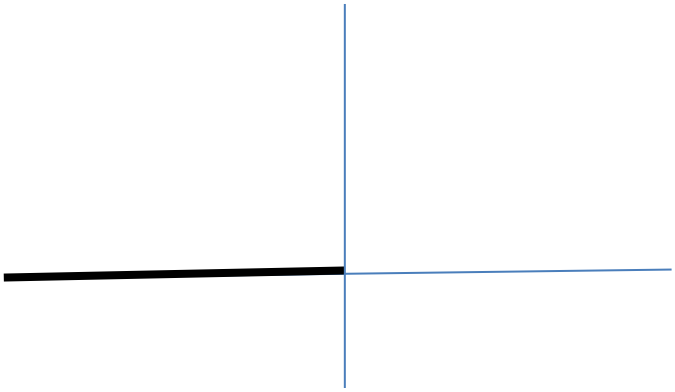
Then the branch



So the logarithm is not defined on the positive real *jumps with a jump of* $i2\pi$, or the argument varies in

$$-\pi < \theta \leq +\pi$$

Then the branch



So the logarithm is not defined on the negative real and *jumps with a jump of* $i2\pi$,

we have

$$(w)^{z-1} = e^{(z-1)\log(w)} = e^{(z-1)(log|w|+i\theta)}$$

The points $x = \pm 2ni\pi$ are singularities for the function $\frac{(-x)^{z-1}}{e^{x}-1}$ .taken in the positive sense; the integral $\int_{+\infty}^{+\infty} \frac{z^{w-1}}{e^{z}-1} dz$ is composed of three parts:

a) L'intégrale de $+\infty$ à :

$$\int_{+\infty+i\varepsilon}^{0} \frac{w^{z-1}}{e^{w}-1} dw$$

above the positive real axis where $arg(logw)$ is 0.

b) the integral around the origin 0 :

$$\int_{|w|=\varepsilon} \frac{w^{z-1}}{e^{w}-1} dw$$

means around the circle with radius $r = |w| = \varepsilon$, the $arg(logw)$ changes from 0 to 2π, so to calculate the integral we make $|w| = \varepsilon \longrightarrow 0$, by changing $z = re^{i\theta}$

c) L'intégrale from 0 to $+\infty$:

$$\int_{0}^{+\infty-i\varepsilon} \frac{w^{z-1}}{e^{w}-1} dw$$

Under the positive real axis, $arg(logz)$ is 2π.

then

$a)\ \int_{+\infty+i\varepsilon}^{0+i\varepsilon} \frac{w^{z-1}}{e^{w}-1}dw = \int_{+\infty+i\varepsilon}^{0} \frac{e^{(z-1)(log|w|+0)}}{e^{z}-1}dz = -\int_{0}^{+\infty} \frac{w^{z-1}}{e^{w}-1}dw$

$b)\ \int_{C} \frac{(w)^{Z-1}}{e^{z}-1}dx$, by using the changing $w = e^{i\theta}$ we find but since

$$|(w)^{z-1}| = e^{(\sigma-1)(log|w|-t\theta)} = |w|^{\sigma-1}.e^{(1-\sigma)t\theta} \leq |w|^{\sigma-1}.e^{2\pi t}$$

And $\exists$ positive α such that

$$|e^{w}-1| > \alpha|w|$$

then

$$\int_{|w|=\varepsilon} \frac{w^{z-1}}{e^{w}-1}dw < \int_{|w|=\varepsilon} \frac{|w|^{\sigma-1}e^{2\pi t}}{\alpha|w|}dw$$

Which $\longrightarrow 0$ when $|w| = \varepsilon \longrightarrow 0$.

$$\int_{0}^{2\pi} \frac{w^{Z-1}}{e^{w}-1}d\alpha = 0$$

$c)\ \int_{0-i\varepsilon}^{+\infty-i\varepsilon} \frac{(w)^{Z-1}}{e^{w}-1}dw = \int_{0}^{+} \frac{e^{(z-1)(log|w|+i2\pi)}}{e^{w}-1}dw$

$$= e^{(z-1)i2\pi} \times \int_{0}^{+\infty} \frac{e^{(z-1)log(w)}}{e^{w}-1}dw$$

$$= e^{zi2\pi} \times \int_{0}^{+\infty} \frac{w^{Z-1}}{e^{w}-1}dw$$

Then the two integrals

$$= -\int_0^{+\infty} \frac{w^{Z-1}}{e^w-1} dw + e^{zi2\pi} \times \int_0^{+\infty} \frac{w^{Z-1}}{e^w-1} dw$$

$$\int_{+i\varepsilon+\infty}^{-i\varepsilon+\infty} \frac{w^{z-1}}{e^w-1} dw = (e^{zi2\pi}-1) \int_0^{+\infty} \frac{w^{z-1}}{e^w-1} dw$$

$$= (e^{zi2\pi}-1)\Gamma(z)\zeta(z)$$

then

$$\zeta(z) = \frac{e^{-i\pi z}\Gamma(1-z)}{2i\pi} \int_{+\infty}^{+\infty} \frac{x^{z-1}}{e^x-1} dx$$

Riemann's method

The path C choisi par Riemann in the complex plan $\mathbb{C}$ in the positive sens

$$\int_{C^+} \frac{(-x)^{z-1}}{e^x-1} dx$$

And as before

$$(-x)^{z-1} = e^{(z-1)\log(-x)} = e^{(z-1)(log|-x|+i\theta)}$$

Riemann chose for $logz$ the branch in which $log\ z$ is real for real z, the cut is $]-\infty,0[$ which is $[-\pi,\pi[$, and that means that $(-x)^{z-1}$ is not defined on the real line $]0,+\infty[$.

The log is real means that $x<0$ then $\theta=0$ and

$$-\pi<\theta\leq\pi$$

The C^{+} in the positive sense , does not contain the numbers $x=\pm 2ni\pi$ for which the term $\frac{(-x)^{z-1}}{e^{x}-1}$ n'est pas défini mais seulement le number $x=0$ and

by the symbol $(-x)$ **Riemann** means that the path of integration does not touch the positive real line and as $(-x)=\ e^{(z-1)\log(-x)}$ log a negative is not defined on the positive real line.

thus the improper integral : $\int_{+\infty}^{+\infty}\frac{(-x)^{z-1}}{e^{x}-1}dx$ est composée de trois parties:

a) the integrale from $+\infty$ to ε : $\int_{+\infty+i\varepsilon}^{0}\frac{(-x)^{z-1}}{e^{x}-1}dx$ above the real axis :

b) L'intégrale around the origin 0 : $\int_{|x|=\varepsilon}\frac{(-x)^{z-1}}{e^{x}-1}dx$, $|x|=\varepsilon$, means around the circle with radius $r=\varepsilon$.

c) L'intégrale de ε à $+\infty$: $\int_{0}^{+\infty-i\varepsilon}\frac{(-x)^{z-1}}{e^{x}-1}dx$ au-dessous rela axis

We easily get for the value of this integral

$$(e^{-\pi si} - e^{\pi si}) \int_0^{+\infty} \frac{x^{s-1}}{e^x - 1} dx$$

Making the hypothesis that, in the multiform function

$$(-x)^{s-1} = e^{(s-1)log(-x)}$$

the logarithm of $(-x)$ is determined such that it is real when x is negative.

The variable x is considered here as a complex variable $x = \sigma + it$ and z as a given complex quantity.

Riemann obtained this result as follows: integration in the complex plan is amde

The term $\frac{(-x)^{Z-1}}{e^x-1}$ is not defined at the value $x = 0$, so it has a singularity a the point $x = 0$.

The definite integral : $\int_{\infty}^{\infty} \frac{(-x)^{Z-1}}{e^x-1} dx$ is composed of three parts according to the argument θ

$a)$ integrale from $+\infty$ to 0 : $\int_{\infty}^{0} \frac{(-x)^{Z-1}}{e^x-1} dx$: the argument is $\theta = -\pi$

$$\lim_{\varepsilon \to 0} \int_{+\infty}^{\varepsilon} \frac{e^{z(logx-\pi)}}{x(e^x - 1)}$$

$b)$ the integrale around the origin : $\int_{|x|=\varepsilon} \frac{(-x)^{Z-1}}{e^x-1} dx$: l'argument θ varie de 0 à 2π

$c)$ the integrale from 0 to $+\infty$: $\int_{0}^{\infty} \frac{(-x)^{Z-1}}{e^x-1} dx$: the argument is $\theta = +\pi$

$$\lim_{\varepsilon \longrightarrow 0} \int_{\varepsilon}^{+\infty} \frac{e^{z(logx+\pi)}}{x(e^x-1)}$$

so

$$\int_{+\infty}^{+\infty} \frac{(-x)^{Z-1}}{e^x-1}\,dx = e^{zi\pi} \times \int_0^{+\infty} \frac{x^{Z-1}}{e^x-1}\,d\alpha - e^{-zi\pi} \times \int_0^{+} \frac{x^{Z-1}}{e^x-1}\,d\alpha$$

$$= (e^{zi\pi} - e^{-zi\pi}) \times \int_0^{+\infty} \frac{x^{Z-1}}{e^x-1}\,dx$$

one may take the integral

$$\int_{C^+} \frac{x^{Z-1}}{e^x-1}\,dx$$

By a path the axis of negative real numbers , the argument changes de as in the figure :

Fig.3: another path in $\mathbb{C}$ where $R(z) < 0$

we will have then

$$\boldsymbol{2sin(\pi s)\Pi(s-1)\zeta(s) = i\int_{\infty}^{\infty}\frac{(-x)^{s-1}}{e^{x}-1}dx}$$

The integral being defined as indicated above.

for our integral

$$\int_{\infty+i\varepsilon}^{\infty-i\varepsilon}\frac{(-\mathrm{x})^{\mathrm{Z}-1}}{\mathrm{e}^{\mathrm{x}}-1}dx = (e^{-\pi zi}-e^{\pi zi})\int_{0}^{+\infty}\frac{x^{\mathrm{z}-1}}{\mathrm{e}^{x}-1}dx$$

$$= (\cos(\pi z)-isin(\pi z)-\cos(\pi z)-isin(\pi z))\int_{0}^{+\infty}\frac{x^{\mathrm{z}-1}}{\mathrm{e}^{x}-1}dx$$

$$= -i2sin(\pi z)\zeta(\mathrm{z})\Gamma(\mathrm{z})$$

by multiplying by the number i in the equation one finds as **Riemann** wrote

$$i\int_{+\infty+i}^{\infty-i}\frac{(-\mathrm{x})^{\mathrm{Z}-1}}{\mathrm{e}^{\mathrm{x}}-1}dx = 2sin(\pi z)\Gamma(\mathrm{z})\zeta(\mathrm{z})$$

where $\Gamma(\mathrm{z}) = \Pi(z-1)$. **Edwards** found

$$\zeta(\mathrm{z}) = \frac{\Gamma(\mathrm{z})}{2\pi i}\int_{+\infty}^{+\infty}\frac{(-\mathrm{x})^{\mathrm{Z}-1}}{\mathrm{e}^{\mathrm{x}}-1}dx$$

and he says

*this formula is misstated by the editors of **Riemann**'s works in the notes; they put the factor on the wrong side of the equation.*

In fact in Riemann work published wrote

$$\zeta(\mathrm{z}) = \frac{\pi\Gamma(\mathrm{z})}{2i}\int_{+\infty}^{+\infty}\frac{(-\mathrm{x})^{\mathrm{Z}-1}}{\mathrm{e}^{\mathrm{x}}-1}dx$$

This equation gives now the value of the function $\zeta(s)$ for every complex value of s and tells that this function is uniform, that it is finite for every finite value of s, except for the value $s = 1$, and also that it vanishes when s equals an even negative integer.

Riemann says that he has found a continuation of the series $\sum_{n\geq 1}\frac{1}{n^z}$ to the whole complex plane by the equation trouvée

$$i\int_{\infty+i}^{\infty-i}\frac{(-\mathrm{x})^{\mathrm{Z}-1}}{\mathrm{e}^{\mathrm{x}}-1}dx = 2sin(\pi z)\zeta(\mathrm{z})\Gamma(\mathrm{z})$$

the function ζ is called *uniform* so it is well defined : it is single-valued(or *monogenic*), it takes only one value for every complex number z, that is it is not *multiform* and the definition of a function today in modern terminology a *function* is necessarily *uniform* and well defined as an *application* between two sets , in our case

$$\zeta \ : \ \mathbb{C}/\{+1\} \longrightarrow \mathbb{C}$$

$$\mathrm{z} \longrightarrow \zeta(\mathrm{z}) = \frac{i}{2sin(\pi z)\Gamma(\mathrm{z})}\int_{\infty}^{\infty}\frac{(-\mathrm{x})^{\mathrm{Z}-1}}{\mathrm{e}^{\mathrm{x}}-1}dx$$

but he has his concerns of *multiform function* because unlike real analysis , they occurs much in complex analysis as

$$z^{\frac{1}{n}}, \quad \log(z), \qquad \arctan(z), \ldots$$

And he dealt with them by introducing his famous concept ***Riemann*** *surface* , they are related to each other through relations such as ***Bernoulli*** *formula*(John, 1702)

$$\frac{1}{a} arctang\left(\frac{z}{a}\right) = \frac{i}{2a} log \frac{ai + z}{ai - z}$$

and **Euler**'s (1747)

$$arcsin(z) = -ilog(iz \pm \sqrt{1 - z^2})$$

The equation by **Riemann** is written for $\sigma > 1$

$$\zeta(z) = \sum_{n \geq 1} \frac{1}{n^z} = \frac{i}{2\sin(z\pi)\Gamma(z)} \int_{+\infty}^{+\infty} \frac{(-x)^{z-1}}{e^x - 1} dx$$

Definition

For $\sigma \neq 1$

$$\zeta(z) = \frac{i}{2\sin(z\pi)\Gamma(z)} \int_{+\infty}^{+\infty} \frac{(-x)^{z-1}}{e^x - 1} dx$$

for all complex values car

i) the integral converges for every complex value z because $|x^{z-1}| = o(e^x)$, seen that $\left|\frac{(-x)^{z-1}}{e^x-1}\right|$

ii) for the value $z = 1$ the $\sin(\pi) = 0$ which has a pole at with residue

$$res_{z=1} = 1$$

Because

iii) for $z = -2n$ on a

$$\zeta(-2n) = \frac{i}{2\sin(-2n\pi)\Gamma(-2n)} \int_{+\infty-i0}^{+\infty+i0} \frac{(-x)^{-2n-1}}{e^x - 1} dx$$

but by le theorem of **Cauchy**

$$\int_{C^+} \frac{(-x)^{-2n-1}}{e^x - 1} dx = 0$$

gives : $\forall n \in \mathbb{Z}/\{0\}$

$$\zeta(-2n) = 0$$

the poles de $\Gamma(-2n)$ are eliminated by the zeros de $sin(\pi z)$ for

$$\lim_{z \to -2n} \sin(z\pi)\Gamma(z) < \infty$$

For other methods for continuation of the function to $\mathbb{C}$, see In his book[1], **Titchmarsh** gave 6 other methods of the continuation . The modern methods de power series et de *analycity* is rhetorical, justifies **Riemann**'s method du continuation de la function ζ which is rather mechanical and practical.

Exercises

1) find the value of the definite integral $I(t) = \int_a^b x^t dx$ for all possible real number t.

1) the same with

$$I(t) = \int_{\varepsilon}^{+\infty} x^t dx$$

1) calculate the real definite integral $\int_{-1}^{+1} \frac{1}{x^2} dx$, what about $\int_{-1}^{+1} \frac{1}{x} dx$?

1) calculate the real definite integral $\int_{-1}^{+1} \frac{1}{x} dx$ by putting $x = e^{i\theta}$ ***(Poisson** 1815)*

1) explain why the expression

$$log(z) + log(z) = 2log(z)$$

is incorrect.

1) prove that $\int_0^{+\infty} \frac{x^{z-1}}{e^x - 1} dx$ is convergnte for $\Re(z) > 1$

1) calculate $\int_C \frac{x^{z-1}}{e^x - 1} dx$ where C is the square $(a + ib, -a + ib, -a - ib, a - ib)$

1) evaluate $\int_C \frac{x^{z-1}}{e^x - 1} dx$ by the $x = e^{i\theta}$

1) find : $\int_{+\infty}^{+\infty} \frac{1}{e^x-1}\,dx$

1) calculate : $\lim_{z\to 0} \sin(z\pi)\Gamma(z)$

1) calculate : $\lim_{z\to -2n} \sin(z\pi)\Gamma(z)$

1) calculate : $\zeta(-(2n+1))$

1) calculate : $\zeta(0), \zeta(\frac{1}{2})$

1) give the expression $sinh(t)$ and $cosh(t)$

1) solve : $sinh(t) = 0$ and $cosh(t) = 0$

1) when is $\cos(z)$ real? imaginary?

1) find $cos(it)$ and $sin(it)$

1) calculate $\cos(\pi z)$ and $\sin(\pi z)$

1) solve $\cos(\pi z) = 0$ and $\sin(\pi z) = 0$

1) prove that $cos^2(z) + sin^2(z) = 1$

1) find $|\sin(\pi z)|$ and $|\cos(\pi z)|$

1) prove that $|\cot(\pi z) + i| = \frac{2}{1+e^{2\pi\sigma}}$

1) show that $arcsin(t) = -ilog(iz \pm \sqrt{1-z^2})$

1) where is it true $cos(z) + isin(z) \geq 0\ (< 0)$

1) is a^z uniform ? analytic?

1) show that:

$$\lim_{z\to -2n} \sin(z\pi)\Gamma(z) < \infty$$

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The functional equation of ζ

When the real part of s is negative, the integral, instead of being taken in the positive sense around the domain of such values, may be taken in the negative sense around the domain of all other values that remained.

From the previous equation

$$i\int_{\infty+i}^{\infty-i}\frac{(-x)^{Z-1}}{e^x-1}dx = 2sin(\pi z)\Gamma(z)\zeta(z)$$

It remains to **Riemann** now to calculate the integral

$$I(z) = \int_{+\infty-i\varepsilon}^{+\infty+i\varepsilon}\frac{(-x)^{z-1}}{e^x-1}dx$$

In the other side of the domain, i.e. when the integral is taken in the negative sense, or “clockwise”. The first integration is done in this domain when

$$\sigma = \Re(z) > 0$$

and now when

$$\sigma = \Re(z) < 0,$$

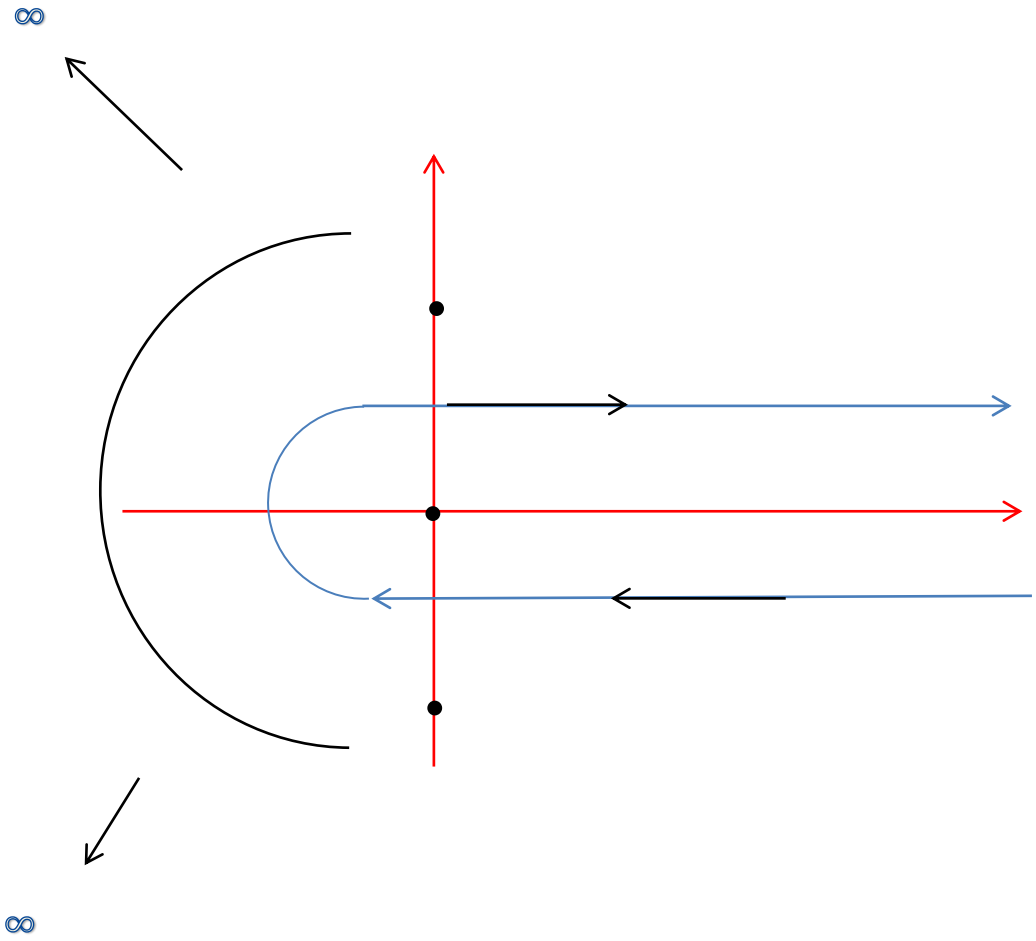


Fig. an integration path on the other side.

witch gives the two integration

$$\int_{C^+} = -\int_{C^-}$$

or

$$\zeta(z) = -\int_{C^-} \frac{(-x)^{z-1}}{e^x - 1} dx$$

for , in this case $\Re(s) < 0$, the integral for values whose module is infinitely big, is then infinitely small .

the integral of the function $\frac{(-x)^{z-1}}{e^x-1}$ for infinitely big values of x tends towards 0 (infinitely small) . Because, for a given z, e^x is infinitely bigger than x^z , which is translate by the limit :

$$\lim_{|x|\to+\infty} \frac{x^{z-1}}{e^x} = 0$$

But, inside this domain, the function under the integration sign does not become discontinuous only if x is equal to a whole multiple of $\pm 2\pi i$, and the integral then, is equal to the sum of integrals taken in the negative sense around these values.

The residues theorem

In this domain there are an *infinity* of singularities of the function under integration sine i.e. the function $\frac{(-x)^{z-1}}{e^{x}-1}$, which are the numbers $x = +i2n\pi$ and $x = -i2n\pi$, these singularities are poles, the integration is done around the circles C_n

$$\int_{C^-} \frac{(-x)^{z-1}}{e^{x}-1} = \sum_{n=-\infty}^{n=+\infty} \int_{C_n} \frac{(-x)^{z-1}}{e^{x}-1}$$

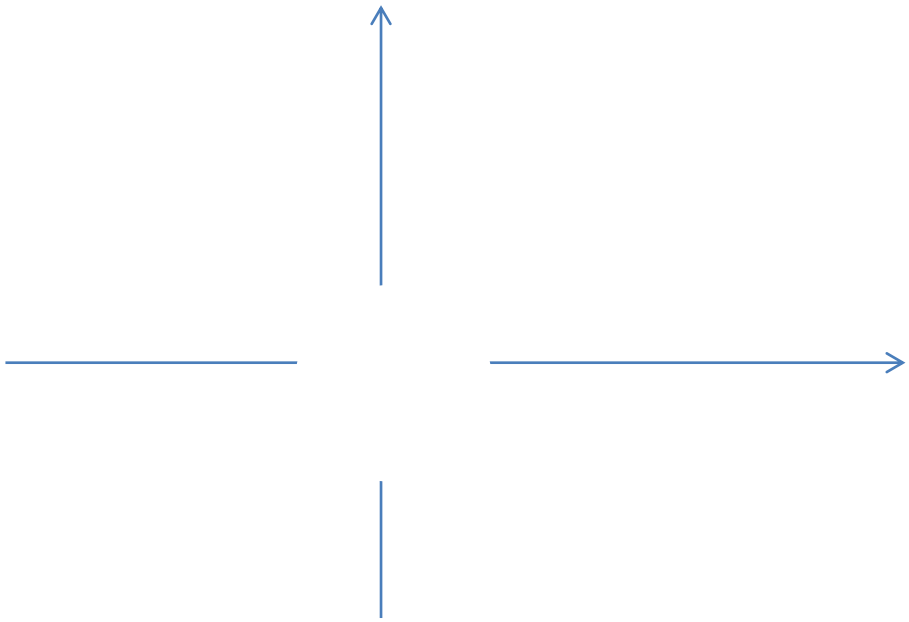


Fig. the poles of the function $\frac{x^{z-1}}{e^{x}-1}$

We simple by a finite domain then we to ∞

Riemann calculated the integrals

$$\int_{C_n} \frac{(-x)^{z-1}}{e^x - 1}$$

through C_n around the poles. By using a known result of complex analysis named *residues theorem* , aresult due to **Cauchy** (1825) who stated it in his " *mémoire*… " :

Residues theorem

Let f defined inside the domin C with poles $z_1, \dots, z_n$ then

$$\int_C f(z)dz = 2\pi i \sum_i res_i$$

The *residue* is the number a_{-1} in the **Laurent** development which we have already mentioned about power series:

$$f(z) = \cdots + + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \cdots$$

Because when we integrate f through C

$$\int_C f(z)\,dz = \dots + \int_C \frac{a_{-2}}{(z-z_0)^2}\,dz + \int_C \frac{a_{-1}}{(z-z_0)}\,dz + \int_C a_0\,dz + \int_C a_1(z-z_0)\,dz + \dots$$

all the term vanishes excepts the term $\int_C \frac{a_{-1}}{(z-z_0)}\,dz$ which gives

$$\int_C \frac{a_{-1}}{(z-z_0)} dz = a_{-1}$$

Thus a residue is considered as the " remaining " or "the rest" of the series after integration.

But the integral relative to the value $n2\pi i$ égale $(-2\pi i)^{s-1}(-2\pi i)$;

That means that the residue for each singularity is

$$2\pi i \times (\pm 2\pi i)^{s-1}$$

In the case where the singularities are simple poles , we have two methods for calculating the number a_{-1} :

Theorem

Let f is a complex function with a singularity z_0 , then:

$a)$ if $f = \frac{g(z)}{z-z_0}$ then $a_{-1} = g(z_0)$

$b)$ if $f = \frac{g(z)}{h(z)}$ then $a_{-1} = \frac{g(z_0)}{h'(z_0)}$

to illustrate the method we will give an example :

calculate $\int_C \frac{1}{z} dz$, where C is a path by a circle, the function $f = \frac{1}{z}$ has a singularity at the point $z = 0$, the residue is simple to calculate : since $g(z)$ is a constant then $g(0) = 1$, which gives

$$\int_C \frac{1}{z} dz == 2i\pi \times 1 = 2i\pi$$

On calcule le résidu de pour $z = 1$

The function $\frac{(-x)^{z-1}}{e^x - 1}$ under the integral has for poles the numbers

$$x = \pm i2n\pi \ (x \text{ here is considered as a complex number}).$$

We have a sum of small circles around the singularities : $\pm i2n\pi$, by noting that $e^{\pm i2n\pi} = 1$

$$res_n = (-x)^{z-1} = (\pm i2n\pi\ )^{z-1}$$

every integral is

$$I(z) = \sum_{n=-\infty}^{n=+\infty} \int_{C_n} \frac{(-x)^{z-1}}{e^x - 1} = 2\pi i \sum_{n=-\infty}^{n=+\infty} (i2n\pi\ )^{z-1}$$

from

$$I(z) = i \sum_{n=-\infty}^{n=+\infty} (i2n\pi\ )^{z-1}$$

$$= i(2\pi\ )^{z-1} \sum_{n=-\infty}^{n=+\infty} (in)^{z-1}$$

$$= i(2\pi)^{z-1} \sum_{n=-\infty}^{n=+\infty} (n)^{z-1} [(-i)^{z-1} + i^{z-1}]$$

now

$$[i^{z-1} + (-i)^{z-1}] = \frac{1}{i}[i^z - (-i)^z]$$

Since

$$(i)^z = e^{z\pi/2} \quad and \quad (-i)^z = e^{-z\pi i/2}$$

at last

$$\mathrm{I(z)} = i(2\pi)^{z-1} 2sin(\frac{\pi z}{2}) \sum_{1}^{+\infty} \frac{1}{n^{z-1}}$$

Riemann wrote slightly different

we get then

$$\boldsymbol{2sin(\pi s)\Pi(s-1)\zeta(s) = (2\pi)^s \sum n^{s-1}[(-i)^{s-1} + i^{s-1}]}$$

since **Riemann** found

$$2sin(\pi s)\Pi(s-1)\zeta(s) = i\int_{\infty}^{\infty} \frac{(-x)^{s-1}}{e^x - 1} dx$$

or

$$2sin(\pi s)\Pi(s-1)\zeta(s) = i2\pi i \sum_{n=-\infty}^{n=+\infty} (i2n\pi\ )^{z-1}$$

$$= -(2\pi\ )^{z} \sum_{n=-\infty}^{n=+\infty} (n)^{z-1}$$

$$= i(i2\pi\ )^{z-1} \sum_{n=-\infty}^{n=+\infty} n^{z-1}[(-i)^{z-1} + i^{z-1}]$$



And

$$2sin(2z)\zeta(z)\Gamma(z) = (2\pi)^{z} \sum n^{z-1}[(-1)^{z-1} + i^{z-1}]$$

and the infinite series

$$\sum_{n=1}^{+\infty} n^{z-1}$$

is $\zeta(1-z)$

that is , a relation between $\zeta(s)$ and $\zeta(1-s)$ …

this is what we called a functional equation, from the precedente equation , we get a relation between $\zeta(z)$ and $\zeta(1-z)$

$$2sin(z\pi)\Gamma(z)\zeta(z) = ii(2\pi)^{z-1}\zeta(z-1)2sin(\frac{\pi z}{2})$$

by remarking that $\sum \frac{1}{n^{z-1}} = \zeta(z-1)$, but from the relation

$$\sin(\pi z) = \frac{\pi}{\Gamma(z)\,\Gamma(1-z)}$$

one gets at last

$$\zeta(z) = 2^z\pi^{z-1}sin(\frac{\pi z}{2})\Gamma(1-z)\,\zeta(1-z)$$

That is named the *functional equation of* ζ, it gives a relation between $\zeta(s)$ and $\zeta(1-s)$. There is no extravagance in this, for instance the function $f(x) = \frac{1}{1+x^2}$ has the functional equation

$$f(\frac{1}{x}) = x^2 f(x)$$

In fact, we see at once the influence of the gamma functional equation on

$$\Gamma(z) = z\Gamma(1-z)$$

Euler(1749) wrote the *functional equation*

$$\frac{\sum_1^\infty (-1)^k n^{k-1}}{\sum_1^\infty (-1)^k n^{-k}} = \frac{-(n-1)!\,(2^k-1)\cos(k\pi/2)}{(2^{k-1}-1)\pi^k}$$

In it we may see the traces of the η function but the series diverges. **Edwards**(p.12) thinks that **Riemann** discovered the functional equation when he wanted to get the needed formula

$$\zeta(2n) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k+1} B_{2k} (2\pi)^{2k}}{2(2k)!}$$

Because it is difficult to find the values $\zeta(2n)$ from his first integral

$$\zeta(\mathrm{z}) = \frac{i}{2\sin(\mathrm{z}\pi)\Gamma(\mathrm{z})} \int_{+\infty-i0}^{+\infty+i0} \frac{(-x)^{z-1}}{e^x - 1} dx$$

Before **Riemann** , and in 1858[10], the German mathematician Oskar **Schlömlich**(1823-1901) suppose the infinite series

$$f(x) = \sum_{n\geq 0} \frac{(-1)^n}{(2n+1)^x}$$

Then

$$\frac{f(1-x)}{f(x)}) = \Gamma(\mathrm{x}) \left(\frac{2}{\pi}\right)^x \sin(\frac{x\pi}{2}).$$

André **Weil** (1906-1998) states in his article[11] that **Riemann'** friend Gotthold **Eisenstein**(1823-1852) knew already the zeta functional equation in 1849, and probably he communicated it to **Riemann**.

From the functional equation

$$\zeta(1-z) = 2^z \pi^{z-1} cos(\frac{\pi z}{2}) \Gamma(\mathrm{z})\, \zeta(z)$$

we can get

[10] In *Zeitschrift für Mathematik und Physik*, Volume III, p.130-132. See also [**Cah**] [**Had196**].
[11] *On Eisenstein's copy of the Disquisitiones*(ASPM, 1989, p.463-469)

$$cos(\frac{\pi z}{2})\Gamma(\mathrm{z}) \ = 2^z\pi^{z-1}cos(\frac{\pi z}{2})\Gamma(\mathrm{z})\ \zeta(z)$$

which **Euler** wrote as

$$\frac{\sum_1^\infty n^{k-1}}{\sum_1^\infty n^{-k}}\sin(k\pi/2) = \frac{-(n-1)!\,(2^k-1)\cos(k\pi/2)}{(2^{k-1}-1)\pi^k}$$

When examining functional equation again

$$\zeta(z) = 2^z\pi^{z-1}sin(\frac{\pi z}{2})\Gamma(1-\mathrm{z})\ \zeta(1-z)$$

It is easy to remark that ζ has zeros which we will call *trivial zeros*, because they are evident. In fact they are simply the zeros de of the term $sin\left(\frac{\pi z}{2}\right)$, the numbers $z = -2n$.

When $z = 2n$, the possible zeros of $\zeta(1-z)$ are canceled by the pole of $\Gamma(1-\mathrm{z})$ that is:

$$\lim_{z\to 2n}\Gamma(1-\mathrm{z})\ sin\left(\frac{\pi z}{2}\right) \text{ is finite}$$

When $z = 2n+1$, the poles of $\Gamma(1-\mathrm{z})$ are canceled by zeros of $\zeta(1-z)$ or otherwise:

$$\lim_{z\to 2n+1}\Gamma(1-\mathrm{z})\ \zeta(1-z) \text{ is finite}$$

for $z = 1$, the point $z = 1$ remains a pole for we have :

$$\lim_{z\to 1}\Gamma(1-\mathrm{z}) = \infty$$

We can also get the trivial zeros of ζ from the formula

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}$$

When $n = 2k$ we have the known values : $B_{2d+1} = 0$. See **Edwards**(p.12)

Which according to the known proprieties of the function Π, may also say that the quantity

$$\boldsymbol{\Pi(\frac{s}{2} - 1)\pi^{\frac{s}{2}}\zeta(s)}$$

remains unchanged when s remplaced by $1 - s$.

one of the know proprieties of the function Γ is that it verify the formulas:

$$\sin(\pi z) = \frac{\pi}{\Gamma(z)\,\Gamma(1-z)}$$

and

$$\Gamma(z) = 2^z \Gamma\left(\frac{z}{2}\right) \Gamma(\frac{z-1}{2}) \pi^{-1/2}$$

the relation by substitution in the equation

$$\zeta(z) = 2^z \pi^{z-1} sin(\frac{\pi z}{2}) \Gamma(1 - s)\, \zeta(1 - z)$$

one finds

$$\zeta(1-z) = 2^z \pi^{z-1} \frac{\pi}{\Gamma(z)\,\Gamma(1-z)} \Gamma(\tfrac{z}{2}) \Gamma(\tfrac{z-1}{2}) \pi^{-1} 2^z\, \zeta(z)$$

which gives

$$\Gamma(\frac{z}{2}-1)\pi^{\frac{z}{2}}\zeta(z) = \Gamma(\frac{1-z}{2}-1)\pi^{\frac{1-z}{2}}\zeta(1-z)$$

As indicated by **Riemann** , or the value of the function

$$\Gamma(\frac{z}{2}-1)\pi^{\frac{z}{2}}\zeta(z)$$

is unchanged if we change z by $1-z$.

we can obtain some values :

$$\zeta(-1) = 2^z \pi^{z-1} sin(\frac{\pi z}{2}) \Gamma(1-s)\, \zeta(2)$$

So

$$\zeta(-(2n+1)) = 2^z \pi^{z-1} sin(\frac{\pi z}{2}) \Gamma(1-s)\, \zeta(2)$$

Exercises

1) calculate the following numbers

i) $(i)^{z-1}$

ii) $(-i)^{-z}$

1) find the singularities and residues of the following functions

i) $\frac{1}{z^2+1}$

ii) $\frac{z+1}{z-1}$

iii) $\frac{e^z}{z^2+1}$

iv) $\frac{1}{z^2+z+1}$

v) $e^{\frac{1}{z}}$

vi) $\frac{e^z}{z^4}$

vii) $\frac{\sin(z)}{z}$

1) show that $tan(x)$ verifies the functional equation

$$f(2x) = [2 + f(2x)\, f(x)]\, f(x)$$

1) prove that $\int_C \frac{a}{(z-z_0)^2} dz = 0$

1) prove that $\int_C (z - z_0) dz = 0$

1) give the complex trigonometric series for $\frac{1}{\zeta\,(z)}$

1) calculate: $\zeta'(z)$ pour $\sigma > 1$

1) evaluate $\int_C \frac{z-1}{e^z-1} dz$ by residue theorem

1) show that $\int_C \frac{1}{e^z-1} dz = 2\pi i$

1) calculate : $\zeta(-(2n+1))$

1) calculate : $\lim_{z\to 2} \sin(\frac{z\pi}{2})\Gamma(1-z)$

1) calculate : $\lim_{z\to 2n+1} \Gamma(1-z)\,\zeta(1-z)$

1) give the equation : $\zeta(1-z) = 2^z \pi^{z-1} cos(\frac{\pi z}{2})\Gamma(z)\,\zeta(z)$ by two methods.

1) show that:

$$\zeta(z) = \pi^{z-\frac{1}{2}} \frac{\Gamma(\frac{1}{2} - z\frac{1}{2})}{\Gamma(\frac{z}{2})}\, \zeta(1-z)$$

1)from the functional equation evaluate $\zeta(2n)$ again

2) show that $\frac{\Gamma(z)}{\Gamma(1-z)} = 2^z \pi^{z-1} tan(\frac{\pi z}{2})$

1) by using functional equation deduce $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$

1) show that for $\sigma < 1$ we have

$$\zeta(z) = \frac{\eta(1-z)}{1-2^z} 2^z \pi^{z-1} sin(\frac{\pi z}{2}) \Gamma(1-z)$$

1) show that for $z \neq$ we have

$$\eta(z) = \frac{1-2^{1-z}}{1-2^z} 2^z \pi^{z-1} sin(\frac{\pi z}{2}) \Gamma(1-z)\, \eta(1-z)$$

9

Proprieties of the function $\Xi(t)$

This propriety of the function engaged me to introduce, in place of $\Pi(s-1)$ ***the integrale*** $\Pi(\frac{s}{2}-1)$ ***in the genral term of the series*** $\sum_{n=1}^{\infty}\frac{1}{n^s}$***, ce qui fournit une expression très commode de la fonction*** $\zeta(s)$

the propriety of function ζ mentioned par **Riemann** is symmetric , namely that it the identity

$$\Pi(\frac{s}{2}-1)\pi^{\frac{s}{2}}\zeta(s)=\Pi(\frac{1-s}{2}-1)\pi^{\frac{1-s}{2}}\zeta(1-s)$$

He intended to isolate the function

$$\Pi(\frac{s}{2}-1)\pi^{\frac{s}{2}}\zeta(s)$$

to which he will give a special name ξ, va introduce **Riemann** pour on va voir que cette commode to write the funtion $\Pi(\frac{s}{2}-1)\pi^{\frac{s}{2}}\zeta(s)$ as an infinite series or the same

$$\zeta(z)=\sum a_n z^n$$

Then to represent it by the use of its roots .

In fact we have

$$\frac{1}{n^s}\Pi(\frac{s}{2}-1)\pi^{-\frac{s}{2}}=\int_0^\infty e^{-n^2\pi s}x^{\frac{s}{2}-1}dx$$

et , par conséquent, si l'on pose

$$\sum_1^\infty e^{-n^2\pi s}=\varphi(x)$$

on a

$$\Pi(\frac{s}{2}-1)\pi^{-\frac{s}{2}}\zeta(s)=\int_0^\infty \varphi(x)x^{\frac{s}{2}-1}dx$$

Si dans l'équation départ, on fait le changement $t=n^2\pi z$, on a

$$\Gamma\left(\frac{z}{2}\right)=\int_0^\infty e^{-n^2\pi z}x^{\frac{z}{2}}\,dx$$

Si on prend la somme deux membres

$$\sum_n \frac{1}{n^z}\Gamma(\frac{z}{2})\pi^{-\frac{z}{2}} = \sum_n \int_0^\infty e^{-n^2\pi s} x^{\frac{z}{2}-1} dx$$

one finds

$$\Gamma(\frac{z}{2})\pi^{-\frac{z}{2}}\zeta\,(z) \quad = \int_0^\infty x^{\frac{z}{2}-1}(\sum_{n\geq 1} e^{-n^2\pi z})dx$$

Because the infinite series $\sum_n e^{-n^2\pi z}$ converges uniformly , which gives

$$\Gamma(\frac{z}{2})\pi^{-\frac{z}{2}}\zeta\,(z) \quad = \int_0^\infty x^{\frac{z}{2}-1}\varphi(x)dx$$

the function theta $\varphi(x) = \sum_0^\infty e^{-n^2\pi z}$ was studied by **Jacobi** , one of the teachers of **Riemann** in Berlin, it is an infinite series that has a *functional equation* :

$$\sum_{n\geq 1} e^{-n^2\pi z} = \frac{\varphi(x)-1}{2}$$

It is a series that converges very rapidly, and linked to elliptic functions

$$f(z) = \varphi(x) = \sum_{n\geq 1} e^{-n^2\pi z} = \vartheta(z)$$

and **Jacobi** is one of the founders of this fascinating theory of elliptic functions. The *theta* functions were also before studied by **Poisson** (1781-1840) , linked to its ***Poisson*** *summation formula*

$$\sum_{n=-\infty}^{n=+\infty} f = \sum_{n=-\infty}^{n=+\infty} f$$

the function φ is originally defined in the general form

$$\varphi(z, z_o) = \sum_{n=-\infty}^{n=+\infty} e^{-(n+z_0)^2 \pi z}$$

where z_o is a given complex number. In the case $z_o = 0$, we have the function $\varphi(x)$.

or, since

$$\boldsymbol{2\varphi(x) + 1 = x^{-\frac{1}{2}}[2\varphi(\frac{1}{x}) + 1]}$$

we have also

$$\boldsymbol{\Pi\left(\frac{s}{2} - 1\right)\pi^{-\frac{s}{2}}\zeta(s) = \int_0^{\infty} \varphi(x)x^{\frac{s}{2}-1}dx + \int_0^{\infty} \varphi(x)x^{\frac{s}{2}-1}dx + \int_0^{\infty} \varphi(x)x^{\frac{s}{2}-1}dx}$$

$$\boldsymbol{= \frac{1}{s(s+1)} + \int_0^{\infty} \varphi(x)(x^{\frac{s}{2}-1} + x^{-(1+z)/2})dx}$$

the theta function verifies the functional equation

$$\frac{1 + 2\varphi(x)}{1 + 2\varphi(\frac{1}{x})} = \frac{1}{\sqrt{x}}$$



which was written by **Jacobi** in his memoir book :
"$Funda\ \ Nova\ \ teoriae\ \ functiorum\ \ elliptcarum$"

(1844) and in $l'Œuvre\ de\ Jacobi$ volume I, p.235. **Jacobi**'s memoir is mentioned much by **Riemann** in his other works.
Mais la dite fonction ne se trouve pas dans cet ouvrage mais dans un autre ouvrage de Jacobi intitulé.

Si on substitute the equality

$$\varphi(x) = \frac{x^{-\frac{1}{2}}}{2}\left[2\varphi\left(\frac{1}{x}\right)+1\right]-\frac{1}{2}$$

in

$$\Gamma(\frac{z}{2}-1)\pi^{-\frac{z}{2}}\zeta\ (z) \quad = \int_0^\infty x^{\frac{s}{2}-1}\varphi(x)dx$$

we find

$$\Gamma(\frac{z}{2}-1)\pi^{-\frac{z}{2}}\zeta\ (z) = \int_0^\infty x^{\frac{z}{2}-1}\left(\frac{x^{-\frac{1}{2}}}{2}\left[2\varphi\left(\frac{1}{x}\right)+1\right]-\frac{1}{2}\right)dx$$

Ou

$$\Gamma\left(\frac{z}{2}-1\right)\pi^{-\frac{z}{2}}\zeta\ (z) = \int_0^\infty x^{\frac{z}{2}-\frac{3}{2}}\varphi\left(\frac{1}{x}\right)dx - \frac{1}{2}\int_0^\infty x^{\frac{z}{2}-1}dx + \int_0^\infty \frac{x^{\frac{z}{2}-\frac{3}{2}}}{2}dx$$

Now I put : $\boldsymbol{s = \frac{1}{2} + it}$

and

$$\boldsymbol{\Pi\left(\frac{s}{2}\right)(s-1)\pi^{\frac{s}{2}}\zeta(s) = \xi(t)}$$

Riemann makes here a change of variable of the form $z = \frac{1}{2} + it$, and thus defined a new function of the real variable t

$$\xi(z) = (z-1)\Gamma(\frac{z}{2}-1)\pi^{-\frac{z}{2}}\zeta\,(z)$$

He has in mind some to get more information from the function that will be apparent later so we will know why he did exactly show the number $\sigma = \frac{1}{2}$.

Such that :

$$\boldsymbol{\xi(t) = \frac{1}{2} - (t^2 + \frac{1}{4})\int_0^\infty \varphi(x)x^{-\frac{3}{4}}cos(\frac{1}{4}tlogx)dx}$$

or more :

$$\boldsymbol{\xi(t) = 4\int_0^\infty d\frac{[x^{\frac{3}{2}}\psi(x)]}{dx}x^{\frac{1}{4}}cos(\frac{1}{4}tlogx)dx}$$

the function $(z-1)\Gamma(\frac{z}{2}-1)\pi^{-\frac{z}{2}}\zeta\,(z)$ has two singularities : 0 and 1. Pour les éliminer on multiplie par $z(z-1)$ ce qui donne

$$z\,(z-1)\Gamma(\frac{z}{2})\,\pi^{-\frac{z}{2}}\zeta\,(z) = 1 + z(z+1)\int_0^\infty \varphi(x)x^{\frac{z}{2}-1}dx$$

In this new reformulation of **Riemann** $\Pi(\frac{z}{2}-1)z$ becomes $\Pi(\frac{z}{2})$. since ζ has a pole he searched for a function that is defined for all complex numbers an *entire function*:

$$\xi(z) = z(z-1)\pi^{-\frac{z}{2}}\,\zeta\,(z)\,\Gamma(\frac{z}{2})$$



Riemann veut obtenir une function defined on the whole of $\mathbb{C}$, the function ξ est dite *entire function*, il a used $\Gamma(\frac{s}{2})$ pour eliminate the r trivial roots of ζ i.e. $z = -2n$.

He makes the change variable $z = \frac{1}{2} + it$ (on va voir pourquoi) to find a function of real variable (may be he thought it was easier to work with than $\xi(z)$) , we have then :

$$\xi(\frac{1}{2}+it) = (\frac{1}{2}+it)(-\frac{1}{2}+it)\pi^{-\frac{\frac{1}{2}+it}{2}}\,\zeta\,(\frac{1}{2}+it)\,\Gamma(\frac{\frac{1}{2}+it}{2})$$

$$= \frac{1}{\frac{1}{2}+it(\frac{1}{2}+it+1)} + \int_0^\infty \varphi(x)(x^{\frac{\frac{1}{2}+it}{2}-1} + x^{-(1+\frac{1}{2}+it)/2})dx$$

$$= 1 + z(z+1)\int_0^\infty \varphi(x)xe^{\frac{it}{2}logx}dx$$

$$= 1 + z(z+1)\int_0^\infty \varphi(x)x(cos(tlogx) + isin(tlogx)dx$$

Then by integration by parts, we find

$$\xi(t) = 1 + z(z+1)\int_0^\infty \varphi(x)x(cos(tlogx)dx$$

or

$$\xi(z) = \xi(1-z)$$

Riemann made a change of variable for ξ seems a little disorienting, for according to **Edwards** (p. 23) it implied **Riemann** to make a mistake in his calculations.

Riemann's change has been rejected by **Landau** (1877-1938) , a great specialist in number theory of the XX century , he wrote instead:

Definition

let

$$\xi(z) = z(z-1)\pi^{-\frac{z}{2}}\zeta\,(z)\,\Gamma(\tfrac{z}{2})$$

and

$$\Xi(t) = \xi(\frac{1}{2} + \mathrm{it})$$

Landau wrote in German language a big treatise entitled "*The Distribution of Prime Numbers* ", as far as I now it has not been translated yet into other languages.

Then the function $\Xi(\mathrm{t})$ is real and we have then

$$\Xi(t) = \Xi(-t)$$

Which means that Ξ is an even function.

this function is finite for all finite values of t and may be developed in the powers of t^2 in a series that converges very rapidly

The function $\xi(t)$ is *finite* , which means that it has no singularities, i.e. it is defined for all complex values , these function *entire function*

the function

$$\xi(t) = 4\int_0^{\infty} d\,\frac{[x^{\frac{3}{2}}\varphi(x)]}{dx}\,x^{\frac{1}{4}}cos(\frac{1}{4}tlogx)dx$$

may be developed in a power series $\sum_n a_n z^n$ of the form

$$cosh(x) = \sum \frac{x^{2n}}{2n!}$$

These function by **Lambert** (1-1) in to because the parametric for *hyperbola* like the are to the circle and for he complex z we have

$$cosh(z) = \frac{e^z + e^{-z}}{2}$$

and

$$sinh(z) = \frac{e^z - e^{-z}}{2}$$

then

$$\xi(t) = 4\int_0^{\infty} d\frac{[x^{\frac{3}{2}}\varphi(x)]}{dx} x^{\frac{1}{4}}(\sum (-1)^n \frac{(\frac{1}{4}tlogx)^{2n}}{2n!})dx$$

or

$$\xi(t) = 4\sum a_n (z-z)^n$$

where

$$a_n = \int_0^{\infty} d\frac{[x^{\frac{3}{2}}\varphi(x)]}{dx} x^{\frac{1}{4}}dx$$

The series converge very rapidly for **Edwards** commented p.18

"..., the statement that the series $\xi(z) = \sum a_n (z - z)^n$ *converges very rapidly is also a statement that it is like a polynomial of an infinite degree--a finite number of terms gives a very good approximation in any finite part of the plane."*

We will see that **Riemann** intends to write the function ξ as a product of its roots like we do with polynomials.

Perron formula :

$$f = 4\int_0^{\infty} dtlogx)dx$$

Exercises

1) prove that

i) $cosh(z = cosh(z) + icosh(z)$

ii) $sinh(z = cosh(z) + icosh(z)$

iii) $\cos(z) = cosh(z) +$

iv) $sin(z = cosh(z) + osh(z)$

1) find the series

$$cosh(x) = \sum \frac{x^{2n}}{2n!}$$

1) find the zeros of the function

i) $cosh(z)$,

ii) $sinh(z)$

1) show that the zeros of $\xi(t)$ all have the propriety : $0 < \sigma < 1$.

1) show that : $\Xi(t) = \xi(\frac{1}{2} + it)$ is real.

1) show that : $\xi(\frac{1}{2} + it) = \xi(\frac{1}{2} - it)$

1) infer then that : $\Xi(t) = \Xi(-t)$ (the function Ξ is even).

1)sketch $\Xi(t)$ graph in the plane $\mathbb{R}^2$, you must calculate $\xi(\frac{1}{2})$ first.

1) show that : $\xi(0) = -\zeta(0) = \frac{1}{2}$

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The roots of the function ξ(z)

Since for a value of s where the real part is greater than 1,

$$log\zeta(s) = -\sum log(1 - p^{-s})$$

remains finite et que ce même fait a lieu pour les logarithmes des facteurs restants de $\xi(t)$, la fonction $\xi(t)$ peut seulement d'évanouir lorsque la partie imaginaire de t se trouve comprise entre $\frac{1}{2}i$ et $-\frac{1}{2}i$.

We return to the fundamental equation of beginning $\zeta(z) = \prod_p (1 - \frac{1}{p^z})^{-1}$. If we take the logarithm of the two members of the **Eulerian** product

$$log\zeta(z) = log \prod_p (1 - \frac{1}{p^z})^{-1}$$

we find

$$log\zeta(z) = -\sum log(1 - p^{-z})$$

Riemann syas that $log\zeta$ remains finite , which means that the function ζ has no zeros in the region presented by the half plane $\mathcal{R}(s) > 1$, otherwise $log\zeta(z)$ becames infinite at these roots. This fact , is by the theorem of infinite products :

Theorem:

If the product

$$\text{if } \prod_n a_n = 0 \text{ then } \exists k \quad a_k = 0$$

also the terms $(1 - \frac{1}{p^z})^{-1}$ cannot vanish because $p^z - 1 \neq 0$.
by another method is prove that des vanish for $\sigma > 1$ is to look the infinite series:

$$\zeta(z) = \sum \quad < \sum | \quad | < \int \quad = \frac{\sigma - 1}{\sigma}$$

The quantity $\frac{\sigma-1}{\sigma}$ is never 0 for $\sigma > 1$.

We now discuss the function

$$log\xi(z) = \log(z(z-1)\pi^{-\frac{z}{2}}\zeta\,(z)\Gamma\left(\frac{z}{2}\right))$$

The terms :

$$log\Gamma(\frac{z}{2})\ ,\ \zeta\,(z)\ ,\pi^{-\frac{z}{2}}\ \ ,logz(z-1)$$

do not vanish.

And also from the functional equation of ζ: if $\zeta(z)$ has no roots then we infer that $\zeta(1-z)$ also has no zeros. the it remains a domain in the form of a ***strip*** du plan complex defindd by

$$0 \leq \mathcal{R}(z) \leq 1$$

this is the famous critical strip:

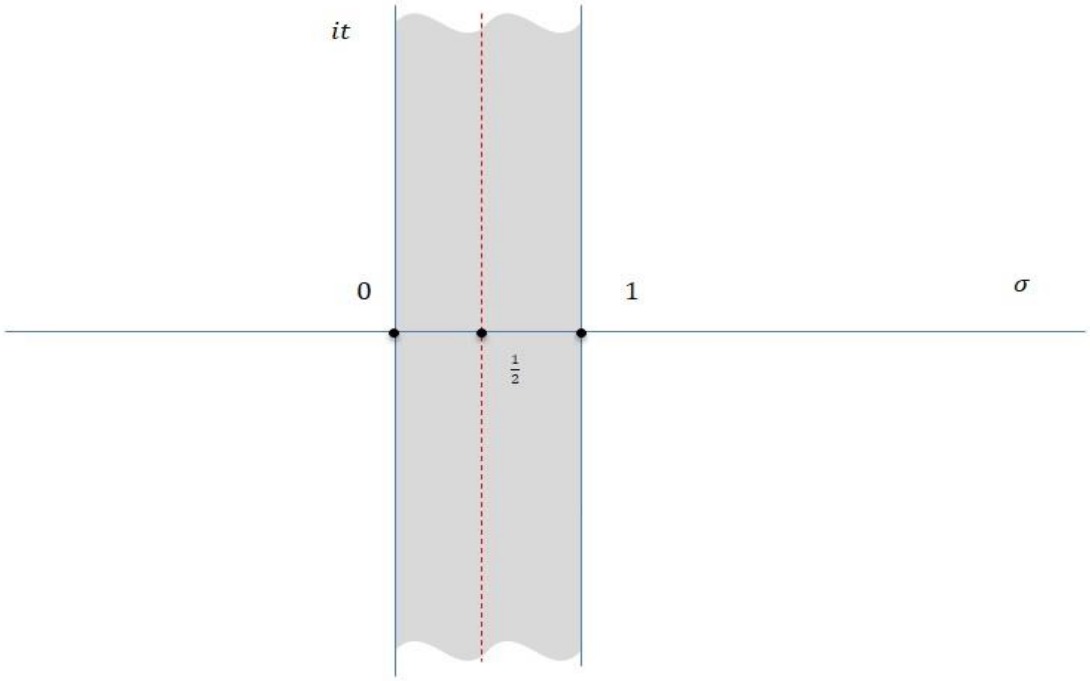


Fig. The critical strip with the critical line.

the function ξ has the same non trivial roots of the function ζ. And since the line $\sigma = \frac{1}{2}$ is an axis of symmetry, and also the point 1/2 , then we write in what follows in the neighborhood of the circle

$$\left|z - \frac{1}{2}\right| \leq R$$

as in the figure:

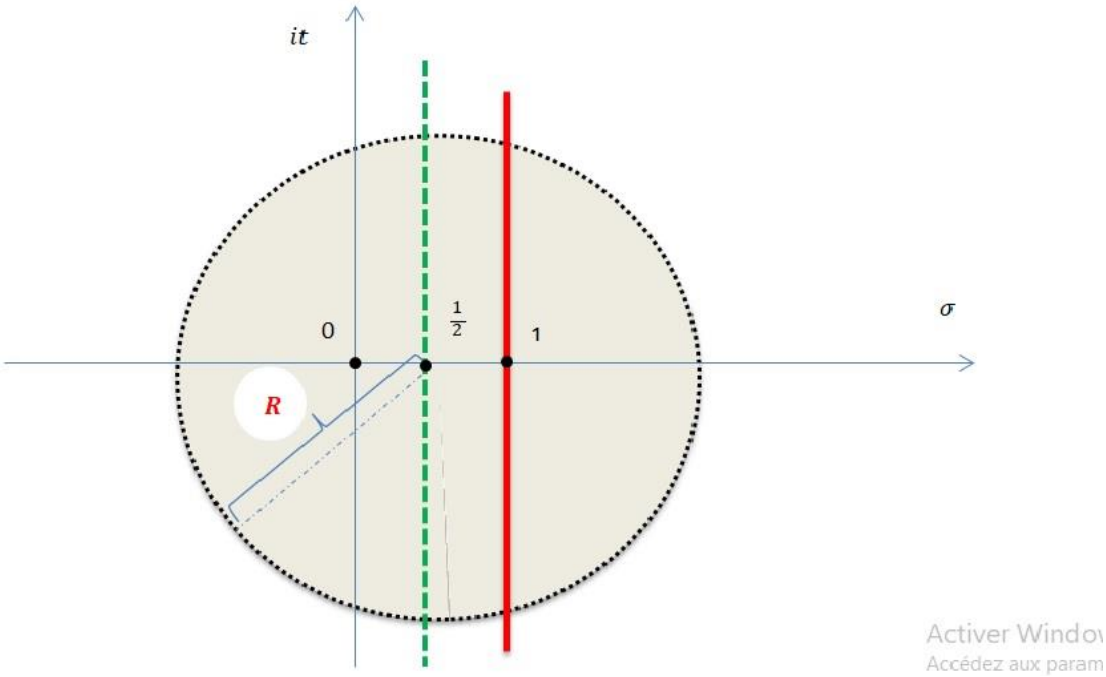


Fig. the circle around the point $\sigma = 1/2$.

since

$$z = \frac{1}{2} + it$$

then

$$z = \frac{1}{2} + i(a + ib)$$

which gives

$$z = \frac{1}{2} - b + ia$$

But we have

$$0 < \frac{1}{2} - b < 1$$

Last the imaginary part b of t verifies

$$-\frac{1}{2} < b < \frac{1}{2}$$

or as **Riemann** wrote

$$-\tfrac{1}{2}i < ib < \tfrac{1}{2}i.$$

Also the function

$$\eta(z) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n^z} = (1 - 2^{1-z})\zeta(z)$$

Has the same roots of ζ.

Theorem:

If a function f is analytic then its zeros are isolated.

Means zeros cannot in a path between two points in C. It is not evident that the function ζ (and therefore ξ too) ait des zeros autres que the zeros $z = -2n$, but **Riemann** has discovered in non-published work that the function ζ has *encrypted roots* (as it would be uncovered by **Siegel** later).

On must remark here to the reader that if the roots sont deux si est un zéro alors est aussi un zéro Pour le négative intéresse σ et non de t , for the Schwarz principle then:

a) if ρ is a root of then selon equation $\xi(z) = \xi(1-z)$, $1-\rho$ is also a zero.

b) if ρ is a root then according to the principe : $\bar{\rho}$ is also a zero, then $\overline{1-\rho}$ is also a zero.

c) the zeros may have multplicities

Alors les zeros non trivial de ξ are

$$\rho, \quad 1-\rho, \quad \bar{\rho} \quad and \quad \overline{1-\rho}.$$

The number of the roots of the equation

$$\boldsymbol{\xi(t) = 0}$$

whose real parts are comprise entre 0 et T is about

$$\boldsymbol{\frac{T}{2\pi} log(\frac{T}{2\pi}) - \frac{T}{2\pi}}$$

To understand what **Riemann** means by saying : ' *same order as* $\frac{1}{T}$ ' we enter in a domain much used by analytic number theory namely : it is very important and useful to solve many problems, to make a comparison between two functions f and g by searching how the function comment f grows compared to g .

we sya that f is of the *same order* as the function g if there is a constant K such that

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} \leq K$$

or by the same meaning $\exists\, K$, $\forall\, x > x_0$ we have :

$$f(x) \leq Kg(x)$$

Ou en la *notation de Landau*

$$f(x) = Og(x)$$

that reads : " f grows no faster than a constant times g ". If f is very small to g then we have

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = 0$$

we write otherwise

$$f(x) = og(x)$$

Riemann means that the number of zeros of ξ entre 0 et T noted $N(t)$ is approximately

$$\frac{T}{2\pi} log(\frac{T}{2\pi}) - \frac{T}{2\pi}$$

avec *un erreur relatif* de l'ordre de $\frac{1}{T}$, ce qui veut dire par la notation de Landau

$$N(t) = \frac{T}{2\pi} log(\frac{T}{2\pi}) - \frac{T}{2\pi} + O(\frac{1}{T})$$

ou de même ∃ K tel que

$$\lim_{t\to+\infty} \frac{N(t) - \frac{t}{2\pi} log\left(\frac{t}{2\pi}\right) + \frac{t}{2\pi}}{\frac{1}{t}} \leq K$$

But he has not mentioned the method used pour obtenir ce result at. He simply added

For the integral $\int dlog\xi(t)$ prise le long d'un contour décrit dans le sens positif, comprenant à son intérieur l'ensemble des valeurs de t dont les parties imaginaires sont situées entre $\frac{1}{2}i$ et $-\frac{1}{2}i$ et les partie réelles entre 0 et T, est égale (abstraction faite d'une partie fractionnaire de même ordre que la grandeur $\frac{1}{T}$) to

$$(Tlog\frac{T}{2\pi} - T)i \quad ;$$

Or cette intégrale est égale au nombre des racines de $\xi(t) = 0$ situées dans ce domaine, multiplie par $2\pi i$.

Number of zeros of an analytic function

ther are the that give number of zeros in examples on it, **Riemann** évoque this formula donnant des nombres des zéros d'une fonction meromorphic (ayant un nombre fini de pole) due to **Cauchy**.

Theorem of the argument(18):

let f a meromorphic function, Z the number of its zeros and P the number of its poles, then

$$Z - P = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\, dz$$

Example:

Give the of the polynomial $f(z) = z^2 + z + 1$ in the , if we try now to in another domain

the argument here is $arg\frac{f'(z)}{f(z)}$, it measures the variation of a function f au long du contour C.

We have in this domain a lot of results that gives information on the zeros of an analytic function f , more precisely , we use a result proved quite lately in 1899 :

Jensen theorem(1899) :

Let f be an analytic function in the disc $|z| \leq R$. if f has no zeros on the disc $|z| \leq R$ and has $z_1, \dots, z_n$ zeros inside with $f(0) \neq 0$ then

$$\log\left|f(0) \times \frac{R}{z_1} \dots \times \frac{R}{z_n}\right| == \frac{1}{2\pi}\int_0^{2\pi} log\left|f(R)e^{i\theta}\right| d\theta$$

The number of zeros in a domain D is very related to how a function grows because is the number is big the analytic and derivable cannot very big to return to the number0 in a short interval, so and we have for

Theorem

in any disc of $D \subset \mathbb{C}$ of radius R we have

$$|\xi(z)| \leq R^R$$

Calculons the integrale $\int_C \ dlog\xi(z)$ like **Riemann** has indicated:

$$\int_C \ dlog\xi(z) = \int_C \ \frac{\xi'(z)}{\xi(z)} dz = 2i$$

Where the path C is indicated in the figure

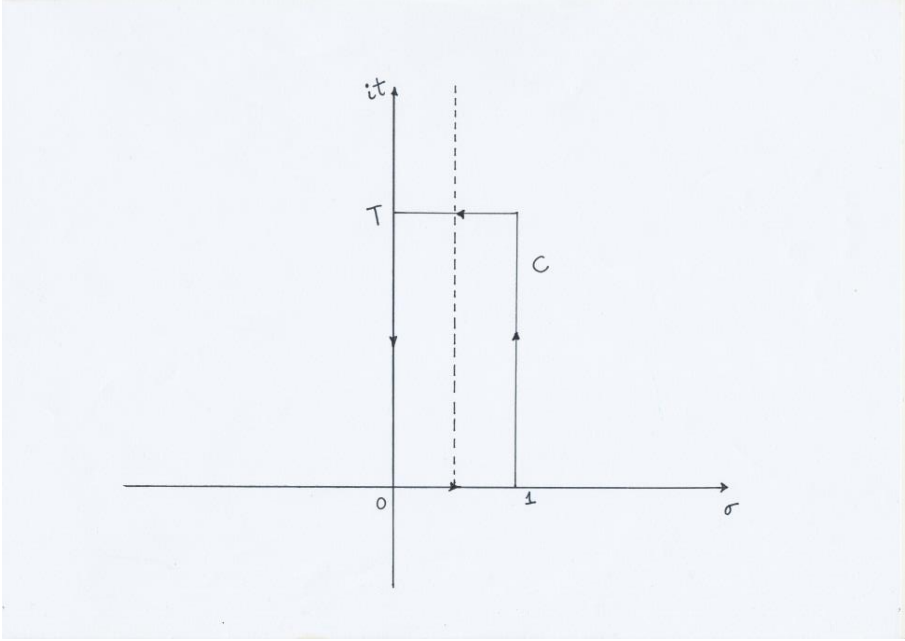


Fig. path of integration

On peut aussi utiliser la function ζ puisque elle a les mêmes zéros non triviaux que ξ

$$N(t) - 1 = \int_C -\frac{\zeta'(z)}{\zeta(z)} dz$$

The values two functions ζ and ζ' at defined point may be well calculated by classical numerical methods like **Euler Maclaurin** formula. Reste le problème : est-ce que les zéros de ζ sont simple ?
On peut utiliser le fait que

Theorem :

If the number of sign changes of the function $\xi(t)$ est égal au nombre des zéros $2N(t)$ then the roots are simple.

this has been proved later by von **Mangolt** in 1905. **Edwards** (p.19) says on **Riemann** proof sketch

His proof of this simply th . Unfortunately he give no hints but if so he was wrong.

So the result is that the error is of the order of 1/T.

Titchmarsh seems to not understand **Riemann** , in fact he says(p.44) **:**

It is usually supposed that 1/T is a mistake for logT; for since N(T) has an infinity of discontinuities at least equals to 1, the remainder cannot tend to zero.

Then also added p. 255

*In the approximate formula for N(T), **Riemann's** 1/T may be a mistake for logT; for, since N(T) has an infinity of discontinuities at least equals to 1, the remainder cannot tend to zero. With this correction, **Riemann's** first statement is Theorem 9.4, which was proved by von **Mangoldt** many years later.*

Edwards commented (p.19):

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***Titchmarsh**, in an unfortunate lapse which he did not catch in the 21 years between the publication of his two books on the zeta function, failed to realize that **Riemann** meant the relative error and believed that **Riemann** had made a mistake at this point.*

In the new edition of ***Titchmarsh*** book in 1985, the commentators wrote (p.286):

***Titchmarsh's** comment on **Riemann's** statement about the approximate formula for $N(T)$ is erroneous. It is clear that **Riemann** meant that the relative error $\{N(T) - L(T)/N(T) \quad is \quad O(T^{-1})\}$.*

Backlund in his to the roots of the as we will see gave a method to ptove Riemann's estimate.

The function $S(t)$:

A function much studied is the function $S(t)$ defined as

$$S(t) = N(t) - 1 = \int_C -\frac{\zeta'(z)}{\zeta(z)}\,dz$$

i.e. it is the number of roots and the limt .

1) find the linear approximation of $f(x) = (4 + x^2)^{\frac{1}{2}}$ at the point $x_0 = 0$

1) solve the equations

i) $cos^2(x) + 3\sin^2(x) - 5\cos(x) = 0$

ii) $6\cos(x) - 8\sin(x) = 0$

1) find the zeros of the function $z^2 + z + 1 = 0$, $cos(z) - 1 = 0$

1) show that $p^z - 1 \neq 0$ for $\Re(z) \geq 1$. What about $\Re(z) > 0$?

1) determinate the roots of the draw them on the complex plan.

$$f(z) = z^2 + 1 \, , f(z) = z^2 - 3z \, , f(z) = z^2 + 2z + 1$$

1) Determinate the roots of the function

i) $z^3 + z^2 + z + 1 = 0$

ii) $3z^2 + 2z + 1 = 0$, draw both of them on the complex plan. What do you remark?

1) let α be a real number , is it true that $\forall z : a^z + (-a)^z = 0$?

1) find the number of zeros of $g(z) = \sqrt{z} + 1$ in $|z| < 2$.

1) give the conjugate of n^{-z} ?

1) let $f(z)$ derivable, show that if f has infinitely many root is the same for $f'(z)$.

1) what about $\int f(z)dz$?

1) show that η has the same roots as in the critical stripe.

1) find all the roots de la function **Dirichlet**'s êta $\eta(z) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n^z}$

1) show that if $f(z) \equiv 0$ then its zeros are not isolated

1) show that : $\sum_{Im\rho \geq 0} \frac{1}{\rho}$ is real

1) show that : $\sum_{Im\rho \geq 0} \frac{1}{\overline{\rho}}$ is real

1) show that if $\sum_{Im\rho \geq 0} \frac{1}{\rho^2}$ converges then $\sum_{Im\rho \geq 0} \frac{1}{\overline{\rho}^2}$ converges too.

1) calculate $\zeta(\frac{1}{2})$, $\xi(1)$

1) show that ζ has roots for $\sigma > 1$ by taking $\frac{1}{\zeta}$.

1) solve in $\mathbb{C}$ the equation $p^z - 1 = 0$.

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1) show that the zeros of ξ all have the propriety $0 \leq \sigma \leq 1$

1) show that η has no roots in the real interval [0,1]

1) show that if ρ is a root then $1 - \rho$ is also a root.

1) show that ρ is zeros , then $\overline{1 - \rho}$ is also a zero.

1) has $log\zeta$ zeros in $\Re(z) > 1$? in $\Re(z) < 1$?

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The Riemann Hypothesis

In fact , we find between these limits a number of real roots , approximately equals to that one ,…

Riemann says that the number of roots of the function of real variable $\xi(t)$ between 0 and t is the same, namely

$$\frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right)-\frac{T}{2\pi}$$

Which means that the number of the complex roots of the function ζ on the critical line is also the same.

Riemann has not given the method by which he obtained this result too, it has never been proved. We will see that other result took decades to be proved, and some

speculated on the meaning of this. **Titschamrsh** said (p.45) **:**

*…But one would like to know what **Riemann** meant by saying 'we find about as many real zeros as there are altogether.' There seems to be no clue to the exact significance of this remark.*

It is that Riemann knew this , or have proved it but no one could know how. It is all in **Riemann** style when writing mathematics, **Edwards** on this this "*Riemann style is difficult*" : []:

*"When **Riemann** make an assertion, it may be something which the reader can verify himself, it may be something which **Riemann** has proved or intent to prove, it may be something which was not proved rigorously until years later, it maybe something which is still unproved and , alas, it may be something which is not true unless the hypotheses are strengthened. …His tragically brief life was too occupied with mathematical creativity for him to devote himself to elegant exposition or to the polished presentation of all his results."*

We can assert this partial result :

Theorem

The function ζ has no roots in the real interval $[0,1]$.

this is easily proved using the eta function

$$\eta(\sigma) = 1 - \frac{1}{2^{\sigma}} + \frac{1}{3^{\sigma}} - \frac{1}{3^{\sigma}}$$

Which positive. Suppose that then but et a is positive and

$$\eta(\sigma) = (1 - 2^{1-\sigma})\zeta(\sigma)$$

Then **Riemann** made another interesting observation:

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It is very probable that all the roots are real .
it would be desirable, without doubt, if we have a rigorous proof to this proposition, however I have for this moment after some futil , for it is recherche côté rapides essais infructueux, parait superflu pour le but immédiat de mon étude.

this fact named by par **Riemann** "*superflue*" , i.e.

all the roots of the function $\xi(\frac{1}{2} + it)$ *, or the same that all the roots of the function* $\Xi(t)$ *are real*

or the same:

all the non-trivial roots de la function $\zeta(z)$ *have all the real part* $\sigma = \frac{1}{2}$

is known today under the name : ***Riemann*** *hypotheses*.
the problem is that no one knows how **Riemann** proved his first statement namely that the number of the complex roots of the function ζ on *the critical line* is

$$\frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi}$$

and perhaps it is this that he lead him to make his big conjecture, as **Edwards** commented p.19 :

"We can only guess, what lay behind this statement, but we do know that it led Riemann to conjecture an even stronger statement, namely, that all the roots lie on $Rez = \frac{1}{2}$. *This is of course the famous Riemann hypothesis. He says that he consider very likely that the roots all do lie on* $Rez = \frac{1}{2}$*", but says that he was not able to prove it (which would seem to imply, incidentally, that he did feel he had rigorous "*

we have already seen the function

$$\eta(z) = \sum_{n\geq 1}^{\infty} \frac{(-1)^{n-1}}{n^z} = (1 - 2^{1-z})\zeta(z)$$

Has the same roots as the zeta function.at this time a lot of false proofs have been presented but no one of them was true.

Why is the Riemann hypothesis so difficult?

The RH is perhaps and without exaggeration, the most difficult problem of all the history of mathematics and this for the following reasons:

i) les number of great genius mathematicians who have attacked it since more than a century without success , and was important them that for example the Last Fermat

theorem..

ii) in hope to crack it mathematicians used fierce technics and amazing machineries in the methods used to attack the problem .

iii) many old problems and also difficult conjectures have been solved like the Grand Theorem de **Fermat** last theorem (1995) and **Poincaré** conjecture (2006) .



iv) the function has in infinite number of zeros.

v) the difficult nature of the function it is known that his function does not very an liner ordinary ODE . the same is true for the gamma function but its roots are regularly spaced. When the American mathematician Hugh **Montgomery** (born in 194) visited **Littlewood** at the end of his life. He told him was working and interested in the **Riemann** hypothesis, **Littlewood** simply said :" *get rid of it*" .

Conrey(2003) says:

a major difficulty in trying to construct a proof of the RH through analysis is that the zeros of L-functions behave so much directly from zeros of many of the special functions were used to seeing in mathematics and mathematical physics. For example, it is known that the zeta-function does not satisfy any differential equation. The functions which do arise as solutions of some of the classical differential equations, such as Bessel functions, hypergeometric functions, ect., have zeros which are fairly regularly spaced. A similar remark holds for the zeros of solutions of classical differential equations regarded as a function of a parameter in the differential equation.

What does RH imply ?

We will see later some propositions in mathematics that are equivalent to the $\boldsymbol{RH}$,

here we treat a different sort of investigations that are , in some sense, go against the truth of RH. To disprove any proposition in mathematics, we work by contraposition:

$$(P \implies Q) \iff (\neg Q \implies \neg P)$$

Titchmarsh says p.76 **:**

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There are two reasons for investigations of this kind. If the Riemann hypothesis is true, it will presumably be proved someday. These theorems will then take their place as an essential part of the theory. If it is false, we may perhaps hope in this way sooner or later to arrive at a contradiction. Actually the theory, as far as it goes, is perfectly coherent, and shows no signs of breaking down.

Sarnak says :

The Riemann hypothesis is the central problem and it implies many, many things. One thing that makes it rather unusual in mathematics today is that there must be over five hundred paper-someday should go and count- which start "Assume the Riemann hypothesis", and the conclusion is fantastic. And those would then become theorems…With this one solution you would have proven five hundred theorems or more at once.

Hardy–Littlewood formula

Hardy and **Littlewood** was very closed friends working at the University of Oxford, were the early mathematicians that attacked the $\boldsymbol{RH}$ since the beginning of the XX century. for his PhD theses **Littlewood** even was assigned to solve the problem. The two great mathematicians have passed a good part of their lives to love it and to search for the proof. thus they were both responsible of the first big results on the zeta function. Before the **Siegel** discovery of the equation **Hardy** and **Littlewood**

obtained an *approximate functional equation*

$$\zeta(z) = \sum_{n \leq \sqrt{\frac{|t|}{2\pi}}} n^{-z} + \frac{1}{2(2\pi)^z cos(\pi \frac{z}{2}) \Gamma(z)} \sum_{n \leq \sqrt{\frac{|t|}{2\pi}}} n^{z-1} + O(f(t))$$

A very practical, where $f(t) = |t|^{\frac{1}{4}-\sigma}$.

the **Riemann** hypothesis has tormented and vexed mathematicians through the whole XX century. **Hilbert** (1900) in his famous list of problems posed the problem number 8 the proof of the Riemann hypothesis. A hundred years after **Hilbert** famous 23 problems, in 2000 the Clay Institute at Paris a pose a price of one million of dollars for a proof of this very difficult conjecture.

Since **Riemann** did not prove that the statement was true and some mathematicians were puzzled and asked for answer to a main question : why did he think that it was very *probable* that his hypotheses was true ?, perhaps because he knew or proved (no one knows how) that *the amount of roots on the critical line* is the same as in the critical strip namely

$$\frac{T}{2\pi} \log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi}$$

And then he heuristically his term "very probable". **Edwards** speculated on this subject in Chapter 7 of his book []:

"We can never know what led Riemann to say it was "probable" that the rootsρ all lie on the line $Res = \frac{1}{2}$*, and the contents of this chapter show very clearly how foolhardy it would be to try to say what mathematical ideas may have lain behind this statement. Nonetheless it is natural to try to guess what have let him to it, and I believe that the Riemann-Siegel formula gives some grounds for a plausible guess."*

Edwards then has given personnel view one can find it p.

We can how the function vary on the line $\sigma = \frac{1}{2}$, as we did before in figure 3 and in this time we put $\sigma = \frac{1}{2}$ and we get a 2 dimensions graph

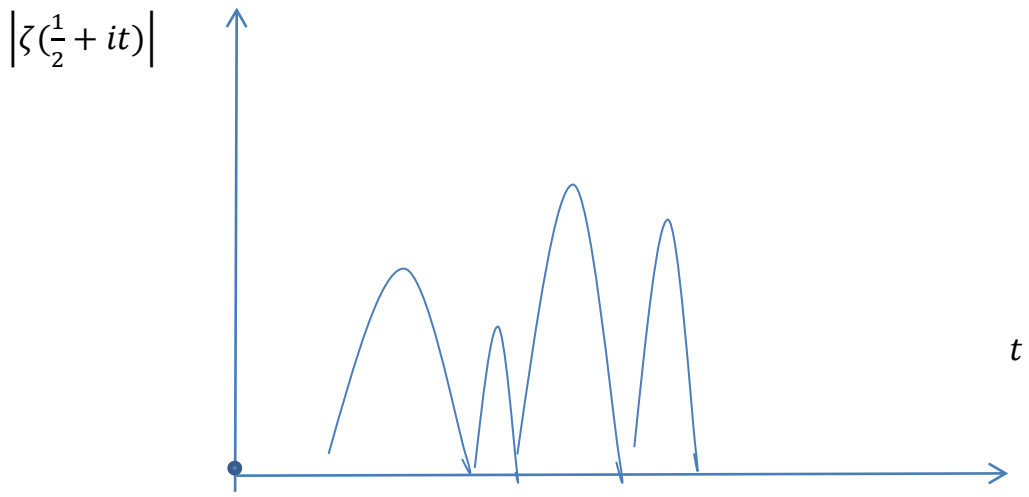


Fig. $\left|\zeta(\frac{1}{2}+it)\right| = f(t)$

Exercises

1) Show that

$$\mathrm{Z(t)} = e^{i\vartheta(t)}\zeta(\frac{1}{2}+it)$$

1) show that : $|Z(t)| = \left|\zeta(\frac{1}{2} + it)\right|$

1) show that

$$\zeta^{(k)}(z) = (-1)^k \sum_{n\geq 2} \frac{log^k(n)}{n^z}$$

Why the sum is from 2 and not from 1? Can you the radius of convergence of this infinite series?

1) show that the **Riemann** hypothesis is equivalent to the condition :

$$\sum_{n\geq 1} \frac{n^{-\varepsilon}(-1)^{n-1}}{\sqrt{n}} \cos(tlogn) = 0 \quad \text{then} \quad \sum_{n\geq 1} \frac{n^{\varepsilon}(-1)^{n-1}}{\sqrt{n}} \cos(tlogn) \neq 0$$

or same

$$\sum_{n\geq 1} d_n \frac{(-1)^{n-1}}{\sqrt{n}} \cos(tlogn) = 0 \quad \text{then} \quad \sum_{n\geq 1} \frac{1}{d_n} \frac{(-1)^{n-1}}{\sqrt{n}} \cos(tlogn) \neq 0$$

1) show that if the RH is true then : $\rho = 1 - \rho, \quad \bar{\rho} = \overline{1 - \rho}$.

1) show that RH is equivalent to : all the roots of the function $\Xi(t)$ are real .

Results on the roots of ζ

Even if mathematicians have not succeed in proving the **Riemann** hypotheses, they have obtained a lot of results concerning the roots of the **Riemann** zeta function . Also **Riemann** could show that

$$\sum_{Im\rho\geq 0}\frac{1}{\rho}=\ \frac{1}{2}\gamma+\frac{1}{2}log\pi+1-log2\pi$$

or exactly :

Theorem

consider the non-trivial roots of ζ (i.e. ρ and $1-\rho$) then for $\varepsilon>0$

$$\sum_{\rho}\frac{1}{|\rho|}=+\infty \qquad , \qquad \sum_{\rho}\frac{1}{|\rho|^{1+\varepsilon}}<+\infty$$

As we know that there are infinitely many roots in the critical strip according to **Riemann** ,the result mean that the roots are quite dense in the critical strip, and the following interesting result is :

Theorem of **Bohr**-Landau (1914) :

There are infinity many roots around the vertical line

$$\sigma = \frac{1}{2}$$

then

Bohr(1887-1951) was a Danish mathematician , brother of famous physicist Niels **Bohr**(18-19) one of the founder of quantum mechanics. This result collaboration with **Landau** was presented the 12 of January 1914 .

On 12th of March the same year, **Hardy** presented a more precising result, in fact **Hardy** has given the first significant result in this for the critical line:

Theorem of **Hardy**(1914) :

There are infinitely many roots of ζ on the critical line $\sigma = \frac{1}{2}$.

Hardy wrote this article in French and one can find it in **Browein**[p.313]. For that **Hardy** used the function $\xi\left(\frac{1}{2} + it\right) = \Xi(t)$, and exactly $\Xi(2t)$.



In fact French mathematician **Cahen**(1865-1941) in his PhD thesis

published in 1894 a work of 91 pages about **Riemann** article and he put forward an integral like the **Mellin** transform. We owe to **Cahen** about the early studies on general **Dirichlet** series

$$f = \sum_{0}^{+\infty} c_n z^n$$

And he introduced

$$e = \int_{a-i+\infty}^{a+i\infty} \Gamma(x) y^{-z} dt$$

So **Hardy** took this as his point of attack, so the integral $\Re(z) = \sigma > 0$ and $a > 0$

$$e^{-z} = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \Gamma(x) z^{-n} du$$

by summation n we find

$$1 + \frac{1}{2\pi i} \int_{a-i+\infty}^{a+i\infty} \Gamma(x) z^{-n} \zeta(2u) du = 1 + 2 \sum_{1}^{+\infty} e^{-n^2 z}$$

So when we put $\sigma = \frac{1}{4}$ and by $\xi\left(\frac{1}{2} + it\right) = \Xi(t)$

$$1 + \sqrt{\frac{\pi}{z}} + \frac{2}{\pi} \int_{-\infty}^{+\infty} \left(\frac{\pi}{z}\right)^{\frac{1}{4}+it} \frac{\Xi(t)}{\frac{1}{4} + 4t^2} dt = 1 + 2 \sum_{1}^{+\infty} e^{-n^2 z}$$

Hardy makes the chnges:

$$\begin{cases} z = \pi e^{i\alpha} \quad where \\ \\ \qquad e^{-z} = \end{cases}$$

he get then

$$1 + \sqrt{\frac{\pi}{z}} + \frac{2}{\pi} \int_{-\infty}^{+\infty} \left(\frac{\pi}{z}\right)^{\frac{1}{4}+it} \frac{\Xi(t)}{\frac{1}{4} + 4t^2} dt = 1 + 2 \sum_{1}^{+\infty} e^{-n^2 z}$$



g

$$c_n = \frac{1}{2\pi!} \int_{-\infty}^{+\infty} \xi\left(\frac{1}{2} + it\right) t^n dt$$

And elliptic we have seen before

Suppose now that $\Xi(2t)$ has always the same sign , thus the function tends to 0 which is impossible.

Titchmarsh (p.) gave many facets of **Hardy's** original argument and simplified the proof. In 1927 **Polya**(1887-1985) proved that the function

$$\Xi(t) = 4\int_{1}^{+\infty} \frac{d}{dx}\left\{x^{\frac{3}{2}}\psi'(x)\right\}x^{\frac{1}{4}}cos\left(\frac{1}{2}tlogx\right)dx$$

has infinity many roots confirming **Hardy's** result.

Following **Hardy** result, the results that concern us here about the zeros of the on the critical line :

– **Hardy** et **Littlewood** en 1921 ont montré que les zéros entre $\frac{1}{2}$ et $\frac{1}{2} + it$ est au moins $\boldsymbol{KT}$

– Herald **Bohr** proved in 1921 that the quantity of zeros between $\frac{1}{2}$ and $\frac{1}{2} + it$ is at least $\boldsymbol{KT}$.

– Selberg proved in 1942 that the amount of roots between $\frac{1}{2}$ and $\frac{1}{2} + it$ is at least $\boldsymbol{KTlog(T)}$

– **Levinson** has showed in 1974, that 1/3 of the zeros are on the critical line.

– **Conrey** proved in 1988, that at least 2/5 of the roots are on the critical line.

Regions without zeros

Subsequent mathematicians' efforts also consisted in finding possible regions without zeros in the critical strip. **Titchmarsh** added p.45 :

*The problem of the zero-free region appears to be a question of extending the sphere of influence of the Euler product (3.1.1) beyond its actual region of convergence; for examples are known of functions which are extremely like the zeta-function in their representation by **Dirichlet** series, functional equation, and so on, but which have no Euler product, and for which the **Riemann** hypothesis is false. In fact the deepest theorems on the distribution of the zeros of $\zeta(z)$ are obtained in the way suggested.*

Of course the first result in this domain was **Hadamard**'s theorem that the function has no zeros on the line $\Re(z) = \sigma = 1$, which implies that form the ζ functional equation function has no zeros on the line $\Re(z) = \sigma = 0$.

a) in 19 **Lavallé-Pousssin** in his as we will see has that has no on the z=1 has later proved the function has no roots in the region

$$\Re(z) \geq 1 - \frac{c}{\log(|t| + 2)}$$

with $c > 0$.

a) in 1922 , **Littlewood** proved the function has no roots in the region

$$\Re(z) \geq 1 - \frac{cloglog|t|}{\log|t|}$$

with $|t| > e^2$.

b)in 19 Hungarian mathematician **Turan**(1-19) there is a domain

$$\Re(z) \geq 1 - \frac{c}{\log(|t| + 2)}$$

such that has no roots.

b) in 19 russian mathematicien N.G. **Tchudakov** (1904-1986) que il n'existe pas domain for $\varepsilon > 0$ and $|t| > 2$

$$\Re(z) \geq 1 - \frac{c}{\log^{\frac{3}{4}+\varepsilon}(|t|)}$$

b)in 1942 russian le mathématicien **Vinogradov**(1-19) que il n'existe no roots pas domaine

$$\Re(z) \geq 1 - \frac{c}{\log^{\frac{2}{3}+\varepsilon}(|t|)}$$

The functions $\zeta^{(k)}(z)$ and their roots

An important question is also held about if the derivatives the function ζ have roots and their exact positions, and also their relation to the $\boldsymbol{RH}$. In the real case ζ we have a known example, namely that the trivial roots of derivate $\zeta'(x)$ from the real function

$$\zeta(x) = 2^x \pi^{x-1} \sin\left(\frac{\pi x}{2}\right) \Gamma(1-x)\zeta(1-x)$$

are between $\alpha_i = -2k$ and $\alpha_i = -2k - 2$.

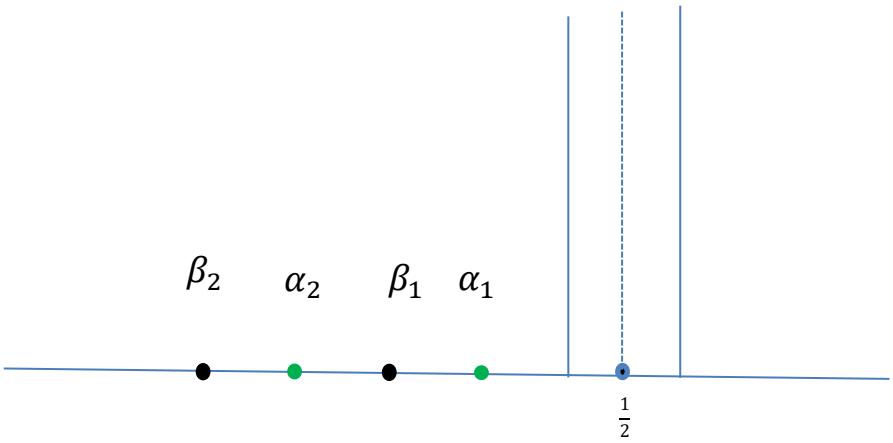


Fig. real roots of ζ and ζ'.

In the complex case one asks if the functions:

$$\zeta^{(k)}(z)$$

has this propriety i.e. the same roots as ζ ? the answer is no!!!! For the numbers $k > 1$ the situation is very complicated and we know now that the function $\zeta^{(2)}(z)$ has a root off the critical strip and more exactly :

$$\zeta^{(2)}(-0.3\ldots + i3.12\ldots) = 0$$

This is not on the critical line.

A classic result about the roots of polynomials derivatives is called **Gauss-Lucas** theorem

Gauss-Lucas Theorem:

Let the complex polynomial $f(z)$ with zeros $z_1, \ldots, z_n$, then the zeros of $f'(z)$ are situated in the convex.

It is just the **Rolle**'s theorem in real case.

Theorem:

Let f a real function that is derivable in ands such that $f(a) = f(b)$ then there exist at least one root of $f'(x) = 0$ between a and b.

As we have seen this result is false for general meromorphic functions. What interests us here is an early significant result on ζ' due to Swiss mathematician Andreas **Speiser** (1885-1970):

Speiser theorem (1934) :

The $\boldsymbol{RH}$ is equivalent to the fact that $\zeta'(z)$ has no roots in : $0 < \sigma < \frac{1}{2}$

We remark here by using the relation to identify roots between roots that since

$$\zeta'(z) = \frac{\eta'(z)}{(1-2^{1-z})} + \eta(z) \times \left(\frac{1}{1-2^{1-z}}\right)'$$

and

$$\eta'(z) = (\Re\eta)_\sigma + i(\Im\eta)_\sigma = (\Im\eta)_t - i(\Re\eta)_t$$

we find that

$$(\Re\eta)_\sigma = 0 \quad , \quad (\Re\eta)_t = 0$$

and thus for a given $\sigma = \sigma_0$, the function $\zeta'(z)$ has the same $\eta'(z)$ roots especially the roots $\left(\frac{1}{2}, \gamma_i\right)$ under RH.

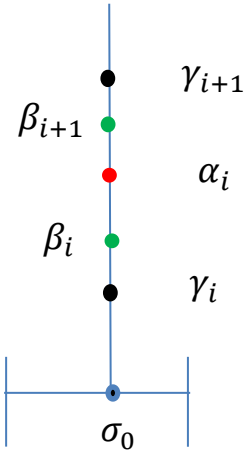


Fig. the critical strip, roots between roots.

Afterwards and in 1974 **Montgomery** and Norman **Levinson** (1912-1975) consolidated and simplified the proof of **Speiser** and established other results like:

Theorem (1974) :

The **Riemann** hypothesis implies that for $k > 1$ the functions $\zeta^{(k)}(z)$ has at most a finite number of non-real roots with $\sigma < \frac{1}{2}$.

Exercises

1) Show that the function has no zeros on the line $\Re(z) = \sigma = 0$.

1) find the roots of f' and show that they verify **Rolle**'s theorem:

i) $f(z) = x^2 + x + 2$
ii) $f(z) = x^3 + x^2 + x + 2$
iii) $f(z) = x^4 + x^3 + x^2 + x + 2$

1) Find the zeros of the function $f(z) = z^4 + 2z^2 + 1$, and show that the zeros of f' verify **Gauss-Lucas** propriety.

i) $f(z) = z^4 + 2z^2 + 1$
ii) $f(z) = z^4 + z^2 + 1$
iii) $f(z) = z^8 + 3z^6 + 4z^4 + 3z^2 + 1$

12

Numerical analysis for sums

Errors

To deal perfectly with what comes on the calculation or roots of the **Riemann** zeta function, we should here give some numerical foundation on numerical calculation and errors. Concerning the sum of series, an early idea comes from the approximation of a definite integral like

$$\int_a^b f(x)dx$$

Where f can be *integrable*, which means the integral is an elementary function. In this case the situation is quite easy. In the case where the integral is not elementary as the known example to the reader

$$\int_a^b \frac{\sin(x)}{x}dx\,, \qquad \int_a^b \frac{dx}{log(x)}\,, \qquad \int_a^b \sqrt{1-\sin(kx)}\,dx$$

and we to understand we by a simple elementary function. So for the first integral we can use power series of the integrand $\frac{\sin(x)}{x}$ and we find with interchange

$$\int_a^b \frac{\sin(x)}{x}dx = 1 + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots \Big|_a^b$$

and to approximate only finite part of terms.

We can perform the inverse: calculate an infinite series by using integration. These methods to approximate definite integrals were known in the XVII century and have three:

−Rectangular method

−Trapezoidal method

−**Simspon** method: to English **Simpson**(1-17).

What is interest us here is that they can used to approximate a sum of finite or infinite series.

Example

Suppose we want to calculate the finite sum

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{10^2}$$

and what know since **Euler** that the infinite series converges to

$$\sum_1^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6} = 1.6449340668\ldots$$

our sum is

$$\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{10^2}=1.$$

so we have measured (i.e. calculated) $S=0.9$. Using now trapezoidal method we find

$$\int_1^{10}\frac{dx}{x^2}\sim\frac{1}{2}\sum_1^{10}f(x_i)(x_i-x_{i-1})+\frac{1}{2}\sum_1^{10}f(x_i)(x_i-x_{i-1})$$

$$=\frac{1}{2}(1+2\frac{1}{2^2}+2\frac{1}{3^2}+2\frac{1}{4^2}+2\frac{1}{5^2}+2\frac{1}{6^2}+2\frac{1}{7^2}+2\frac{1}{8^2}+2\ \frac{1}{9^2}+\frac{1}{10^2})$$

So $\int_1^{10}\frac{dx}{x^2}=\ -\frac{1}{x}\Big|_1^{10}\ =0.9$. So make a between the real measure S of the sum

And the sum numerical integration S_0 we find two numbers $S-S_0=$ and $S_0-S=-$,

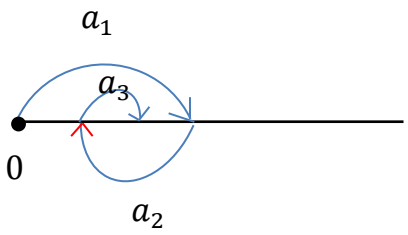
In numerical analysis we always care about the accuracy and look for the methods that give us the least of errors.

Example:

in series called decreasing *alternating series* the error is very easy to calculate , in fact we have always :

$$a_1>S_n$$

the sum of the series is always less the first term, and



The error of the calculation is less than the first term a omitted.

to give an example of the numerical analysis implied her an example:

Example: let find the sum and error of the series $\sum_{n\geq 1}^{\infty}\frac{(-1)^n}{n}$

This is our $-\eta(1)$ where, as we know,

$$\eta(z)=\sum_{n\geq 1}\frac{(-1)^{n-1}}{n^z}=(1-2^{1-z})\zeta(z)$$

so

$$\eta(1)=(0)\times\zeta(1)$$

as a limit. We know from **Leibnitz** criterion that this series converges and we have any problem her because we know the exact sum

$$S=-\sum_{n\geq 1}^{\infty}\frac{(-1)^n}{n}=-\log(2)=-0.765984231$$

the error of the calculation is less than the first a omitted.

$$1 > S_n$$

Also **Leibnitz-Gregory** series for $\frac{\pi}{4}$

$$\sum_0 \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$$

But when we do not algebraically find we approximation, an important is that the sum is less then the first term

$$a_1 \leq S_n$$

or the exactly 1.

The problem is that when one wants to methods:

Error diverges to infinity: The main problem of the error manipulation is the divergence of the terms.

Absolute error

so it is natural if we search for an ***absolute*** error (with no sign) , by using the formula:

$$absolute\ error = \Delta R = |S_0 - S|$$

and find that we have made an error , in our example the error is

$$error = |1 - 0.9| =$$

Relative error

We have seen that for convenience the error must be positive we have absolute error, and the real value between tow *extreme* values then we can write

$$S = S_0 \pm absolute\ error$$

So we have the of accuracy is

$$S_0 - \Delta \mathrm{R} \leq S \leq S_0 + \Delta \mathrm{R}$$

To add another idea to our , in order to know if our approximation is good or not is to compare the error to the measured quantity: and we find naturally the *relative error*

$$e = \frac{e}{S} = \frac{0.9}{1.}$$

And we can think it as percentage $\%$. This is approximation as good the relative error is small. What is , is that we can sometimes find a formula the error and evaluate it.

Also to more we may calcite the approximation by how the contained in and by division we find the relative error

$$arelative\ error = \frac{\Delta R}{}$$

In his **Delavallé-Poussin**(1899) find between the number of primes and the function proposed to approximate his number so he found

$$arelative\ error = \left|\frac{\pi(x) - Li(x)}{Li(x)}\right| = \left|\frac{\pi(x)}{Li(x)} - 1\right| < e^{-\sqrt{clogx}}$$

Which means that the rate by which the $\frac{\pi(x)}{Li(x)}$ goes to 1 is less than for all sufficiently big x.

Euler–Maclaurin formula

We have seen an easy example because we know already the real sum and the terms are easy to calculate , the situation is more difficult when we have no idea about a closed formula of the terms and we have to deal with a function as complicated as the zeta function**.**

Euler already used his formula in the spirit of modern *Numerical Analysis* , to compute

$$\zeta(z) = \sum_{n=1}^{+\infty} \frac{1}{n^{2k}}$$

For $k = 1,2,3, \ldots$ and also to compute his newly discovered (or invented!) $\Gamma(x)$ function for large real numbers x.

The famous formula of **Euler** which approximates a finite sum with an integral can be stated as follows:

Euler-Maclaurin sommation formula :

let f a function then we the sum

$$\sum_{a}^{b} f(n) = \int f(x)dx + \frac{f(1)+f(n)}{2} + \sum \frac{1}{2}\left(f(n)+f(1)\right)$$

The key is to use *integration by parts* to get more term and diminishes the error , a useful technic that **Riemann** himself will use it to avoid problems of divergence of integrals and infinite series.

If there is convergence, if not we have an approximation

$$\sum_{n=a}^{n=b} f(n) \sim \int_{a}^{b} f(x)dx + \frac{f(a)+f(b)}{2} + \sum_{i=1}^{k} \frac{B_{2i}}{2i!} f^{(2i-1)}(x) \Big|_{a}^{b}$$

With an error term of the value

$$R = \frac{1}{(2k+1)!}\int_{a}^{b} B_{2i+1}(x) f^{(2i-1)}dx +$$

Maclaurin also found it independently of **Euler** in 1748. **Euler** it to the infinite series

$$\sum_{n\geq 1}^{\infty} \frac{(-1)^n}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

So when we calculate by this method

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

we find that

Abel summation formula

Gram mentioned a formula due to **Abel** in the sum of series, that give us **Abel**'s summation formula:

Abel(:

let then

$$\sum f(x) = \int f(x) - f(x) + \int_0^{+\infty} \frac{f(x+) - f(x-kt)}{2k(e-1)} dt$$

Then we have

Also

Abel(1826) :

let $a_1, \dots, a_n$ and $b_1, \dots, b_n$ deux suites de nombres réels (or complexes) and

$$S(n) = a_1 + \cdots + a_n$$

Then we have

$$\sum a_n b_n = S(n)\, b_n + \sum S()\,(b_n - b_n)$$

It permits to change the sum of sequences by integrals, which simplifies much the calculation.

Newton and Lagrange methods

For calculating roots of real *continuous* functions, there are two classical methods **Newton**'s and **Lagrange**'s. The principle to locate a root is very simple known to high school student and we will use it all the time in this domain: If f is continuous in the interval $[a, b]$ and

$$f(a)f(b) < 0$$

Then there must be *at least* one root between a and b, this is *the intermediate value theorem*. And if we know that f' has one sign in the interval then the root is *unique*.

Newton method

This a method called also the *method of tangents*. Suppose that we under the previous conditions with the f' and f'' do change in sign, so f may be either convex $(f'' > 0)$ or concave: $(f'' < 0)$. We draw the tangent to $f(b)$ that will the abscissa axis at a point x_0 and we the tangent line at B we find

$$y - f(b) = f'(b)(x - b)$$

And

Lagrange method

This method is an ancient one, known by its Latin name *regula falsi* (false position) , also called ***Lagrange*** *method* or method of *chords* , Newton atngents here we darw chords. in we chord segment f(a), f(b) and the intersection with [] is our root. Then we the oprtion and . this method also *linear interpolation*

Regula falsi:

let f a real function we want to locate a zero:

$$f(\quad) =$$

Laplace method for integrals

to calculate integrals of the form

$$I(t) = \int_0^{+\infty} x^{\alpha} e^{t(a-cx^{\beta})} dx$$

Of the gamma function, and this integral can be written obviously

$$I(t) = e^{t} \int_0^{+\infty} x e^{-cx} dx$$

And it is convergent when : acb.

Laplace method to approximate improper integral of the form

$$I(t) = \int_0^{+\infty} g(x) e^{th(x)} dx$$

And it is consists in *maximum* of $h(x)$. And conditions:

Then an approximation or principal value is

$$I(t) \sim \frac{r}{\beta}\Gamma\left(\frac{\alpha+1}{\beta}\right)e^{at}(ct)^{-\frac{\alpha+1}{\beta}}$$

We consider this

Exercises

1) Suppose that we want to calculate the sum

$$\sum_{1}^{5}\frac{\sin(n)}{n}$$

Use power series then trapezoid method and compare.

1) use trapezoidal method to the sum

$$\frac{1}{2}+\frac{1}{3}\ldots+\frac{1}{7}$$

and find the error, and the relative error.

1) use trapezoidal method to the sum

1) use trapezoidal method to the sum

1) give an approximation to

$$\int_{2}^{x}\frac{1}{\log(t)}dt$$

by integration by parts five times. Is it a good approximation?.

1) calculate by using series:

$$\int_{-1}^{+2} \frac{e^x}{x} dx \quad , \quad \int_{1}^{+2} e^{-x^2}\,\mathrm{dx}$$

$$\int_{1}^{+2} \frac{\cos(x)}{x} dx \quad , \quad \int_{-1}^{+2} \cos(x^2)\, dx \quad , \quad \int_{1}^{+2} \sin(x^2) dx$$

1) if we know that f' has not a unique sign, what we can infer? Show that in this case, it is not necessarily the is not unique. Give an example.

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Calculation of the roots

The search for the roots

After **Riemann** showed that the zeta function has encrypted roots and stated his Hypothesis without proof, the task of mathematicians has been since then to find algorithms to calculate these roots, the roots of the ζ − function or of the ξ −function to see at least if there a counterexample to the hypothesis. Before the discovery of **Siegel** that we will talk about, Danish mathematicians **Gram** (1850-1916) took this seriously and published in 1903, the values of 15 roots of the function , all of which have the real part equal to $\frac{1}{2}$:



$$\rho_1 = \frac{1}{2} + i14.134$$
$$\rho_2 = \frac{1}{2} + i21.022$$
$$\rho_3 = \frac{1}{2} + i25.010$$
$$\rho_4 = \frac{1}{2} + i30.424$$
$$\rho_5 = \frac{1}{2} + i32.935$$
$$\rho_6 = \frac{1}{2} + i37.586$$

$$\rho_7 = \frac{1}{2} + i40.918$$
$$\rho_8 = \frac{1}{2} + i43.327$$
$$\rho_9 = \frac{1}{2} + i48.005$$
$$\rho_{10} = \frac{1}{2} + i49.773$$
$$\rho_{11} = \frac{1}{2} + i52.8$$
$$\rho_{12} = \frac{1}{2} + i56.4$$
$$\rho_{13} = \frac{1}{2} + i59.4$$
$$\rho_{14} = \frac{1}{2} + i61.1$$
$$\rho_{15} = \frac{1}{2} + i65.0$$

His calculations for the imaginary part T were not very accurate, but the importance here is to know that in order to prove the RH for a non-trivial zero ρ_i, one must accomplish two tasks:

Indication

−show that the zero ρ_i has a real part $\sigma = \frac{1}{2}$
−if two roots (ρ_{i-1}, ρ_i) are found to verify RH , then one has to show that there are no other zeros between them.

to calculate these zeros , mathematicians have been using the easy looking equation found by **Riemann**

$$\xi(z) = z(z-1)\pi^{-\frac{z}{2}}\,\Gamma\left(\frac{z}{2}\right)\zeta\,(z)$$

So in the critical strip, the roots of ξ are the same roots of ζ, then when we put

$$\xi(\frac{1}{2}+it)=(\frac{1}{2}+it)\left(\frac{1}{2}+it-1\right)\pi^{-\frac{\frac{1}{2}+it}{2}}\Gamma\left(\frac{\frac{1}{2}+it}{2}\right)\zeta\left(\frac{1}{2}+it\right)$$

we find after simplification

$$\xi\left(\frac{1}{2}+it\right)=[-|\alpha|]e^{i\vartheta(t)}\zeta(\frac{1}{2}+it)$$

where α is a ***negative number*** (i.e. real!!) and

$$\vartheta(t)=\Im\left(\log\Gamma\left(\frac{\frac{1}{2}+it}{2}\right)\right)-\frac{t}{2}\log(\pi)$$

so, as the function ξ is analytic (then continuous) , in order to find, or precisely to "*locate*" a zero , we study the changing of sign from negative to positive or from positive to negative of the function $\xi\left(\frac{1}{2}+it\right)$, as we know that $\xi\left(\frac{1}{2}+it\right)$ is real.

This is very evident because since $-|\alpha|$ is always negative, then the sign of $\xi\left(\frac{1}{2}+it\right)$ is the same sign of $e^{i\vartheta(t)}\zeta(\frac{1}{2}+it)$. So the term on which we concentrate our efforts is :

indication :
To locate a root , we study the sign of :

$$\mathrm{Z(t)}=e^{i\vartheta(t)}\zeta(\frac{1}{2}+it)$$

Towards this aim, **Gram** first used his own original methods before returning to the less sophisticated classical methods : **Euler**-**Maclaurin** formula . So **Gram** used

formulas to *compute* two functions in the critical strip : $0 < \sigma < 1$:

$*$) The first is the function $\vartheta(t)$ which contains the function gamma $\Gamma(z)$ or exactly $\Gamma\left(\frac{\frac{1}{2}+it}{2}\right)$, which is known to be well approximated by **Stirling** series:

$$\log\Gamma(z) = \left(z + \frac{1}{2}\right) logz - z + \cdots R_k$$

with an error formula R_k due to **Stieltjes**[12] (181-) in 1882

$$R = \int_0^{+\infty} \frac{B}{} dx$$

Then find the imaginary part.

$**$) The second term to be computed is the function $\zeta\left(\frac{1}{2} + it\right)$.

The function $\zeta\left(\frac{1}{2} + it\right)$

But how can we evaluate the ζ -series i.e. $\sum_1^{+\infty} n^{-z}$ that does not converge when : $0 < \sigma < 1$?

Gram mentioned the work of French mathematician **Hermite** (18-18) of 1885: form

$$\Gamma(z)\zeta(z) = \int_0^{+\infty} \frac{x^{z-1}}{e^x - 1} dx$$

[12] We will also see the name of **Stieltjes** of the **Riemann** hypothesis.

Then we can write the previous integral as: $\int_0^{+\infty} = \int_0^1 + \int_1^{+\infty} = I_1 + I_2$. Where

$$I_1 = \frac{1}{z-1} - \frac{1}{2z} + \frac{B_1}{2(z+1)} - \frac{B_3}{4(z+3)} + \cdots$$

B_{2k+1} are the **Bernoulli** numbers(we know that $B_{2k} = 0$) , and

$$I_2 = \int_0^{+\infty} \frac{1}{x(e^x - 1)} (1 + \frac{z}{1}\log(x) + \frac{z^2}{2}\log^2(x) + \cdots)dx$$

Although the integral $I = I_1 + I_2$ has no sense for $\Re(z) \leq 1$ but for

to solve this **Gram** used the formula

$$\zeta(z) = \lim_{n=\infty} [\sum_1^n n^{-z} - \frac{n^{1-z}}{1-z}]$$

Or same

$$\zeta(z) - \frac{1}{z-1} = \sum_1^{+\infty} n^{-z} - \int_1^{\infty} x^{-z}dx$$

And this gives an approximation for ζ the in $0 < \sigma < 1$. but from

$$\mathrm{Z(t)} = e^{i\vartheta(t)}\zeta(\frac{1}{2} + it)$$

We can write

$$\zeta\left(\frac{1}{2}+it\right)=\mathrm{Z(t)}\cos\left(\vartheta(t)\right)-i\mathrm{Z(t)}\sin(\vartheta(t))$$

The function $\vartheta(t)$

We have seen that $\vartheta(t)$ as imaginary part and defined as

$$i\vartheta(t) \;=i\Im\left(\log\Gamma\left(\frac{\frac{1}{2}+it}{2}\right)\right)-\frac{t}{2}\log(\pi)$$

so to compute the term $\log\Gamma\left(\frac{\frac{1}{2}+it}{2}\right)$ using **Stirling** series:

$$\log\Gamma(z)=\left(z+\frac{1}{2}\right)logz-z+\cdots R_k$$

we find

$$\vartheta(t)=$$

and after simplification

$$\vartheta(t) = \frac{t}{2} log \frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + \frac{1}{48t} + \frac{7}{5760t^3} + \cdots$$

So the problem is to evaluate the error.

Edwards complains:

…This show that there is a limit to the degree of accuracy with which one can evaluate S by extending the above process , and that this limit is roughly determined by the minimum value of …

for the second task **, Gram** could prove that the roots found are the only ones in the interval $0 \leq T \leq 50$ by : he consider a product representation of $\xi\left(\frac{1}{2} + it\right)$ by the roots $\frac{1}{2} + i\alpha$ we will see later proposed by **Riemann**

$$\xi\left(\frac{1}{2} + it\right) = \xi\left(\frac{1}{2}\right) \prod_{\alpha} (1 - \frac{t^2}{\alpha^2})$$

and by log and **Taylor** representation of $log(1 - x)$ we find for $\Re(\alpha) > 0$

$$\log\xi\left(\frac{1}{2} + it\right) = \log\xi\left(\frac{1}{2}\right) - (\sum \frac{1}{\alpha^2})t^2 - (\sum \frac{1}{\alpha^4})t^4 - (\sum \frac{1}{\alpha^6})t^6 - \cdots$$

this expression may be used to numerically calculate the and the sign and that . the series converges very rapidly and few first term can tell us information about its sign.

Gram points:

During his work on the roots of **Gram** a law and it is between roots: if is a root then . modern as we see that the falsity of the RH and we will a by

Backlund methods

Then , Finnish mathematician Ralf Josef **Backlund** (1888-1949), continued around 1912-1915 the work of **Gram** and got some original results, he proved that the roots of verify the $\boldsymbol{RH}$ up to $T = 200$, he invented a method that enabled him to compute for certain values of T the number of roots up to T. He used Riemann estimate and complex integration in the complex plan so he

$$N(t) = \frac{1}{2\pi i}\int_{C}\frac{\xi}{\xi}d$$

To find that

So the number of roots stake a simpler form

$$N(t) = +1 + \frac{1}{2\pi i}\int_{C}\frac{\zeta}{\zeta}d$$

He also by the method proved again the estimate for the roots in $0 < T$, proved already by von **Mangoldt** that the number of roots is approximately

$$\frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi}$$

Hutchinson and Gram law



American John Irwing **Hutchinson** (1867-1935), proved in 1925 that the roots of the function verify the $\boldsymbol{RH}$ up to $T = 300$,

.

And he the gram points g_n to thus the sign of found a rule

There is always, except for some exceptions, a root between two ***Gram*** *points or the Gram points and the roots separate each other*

Turing(19-195).

Exercises

1) let ρ and $1-\rho$, the non-trivial roots of ζ .

1) complete the calculation missed above to find

$$\xi\left(\frac{1}{2}+it\right) = [-|\alpha|]e^{i\vartheta(t)}\zeta(\frac{1}{2}+it)$$

Find α and be sure that is always negative.

1) t is real, show that $Z(t)$ is real.

1) show that there is no zeros on the line $\sigma = 0$ (from **Hadamard** theorem).

1) calculate

$$\left|\frac{x^{\rho}}{log\rho}\right|$$

1) find the radius de convergence de R de la series $\sum_1(3+(-1)^n)^n z^n$

1) give thes eries: $\frac{1}{1+z}$

1) give **Laurent** series for : $(z-1)cos(\frac{1}{z+2})$

1) whose result is better **Tchukadov's** or **Vinogradov's**?

1) make now a comparison between the results of Lavalle Poussin, Turan, Tchucadov and Vinogradov .

13

Siegel's discovery

Anyone is naturally justified to ask the question :

why did **Riemann** exactly chose $\sigma = \frac{1}{2}$?

some people believed that **Riemann** was perhaps endowed with a supernatural vision , others believed that his choice for the quantity $\frac{1}{2}$ was arbitrary !, or only for the seeking of symmetry but as **Siegel** will say

*The legend that **Riemann** found the results of his mathematical work" highly general" ideas without requiring the formal tools of analysis, is probably no longer as widespread as in **Klein** lifetime*[13].



But in 1926 **,** German **Bessel-Hagen**(18-19) verifying **Riemann** *Nachlass* at the University Library of Göttingen , he told known German mathematician Carl **Siegel** (1896-1981) that he discovered some jottings of **Riemann** he could not decipher , . **Siegel** examined it carefullyand made an account *on Riemann's Nachlass regarding analytic number theory*"(1932)", which contains two things:

i) that **Riemann** discovered a very good asymptotic formula for $Z(t)$ and by it **Riemann** could calculated some of the non-trivial (encrypted) zeros of the function ζ which are.

[13] Translated by E. **Barkan** and D. **Sklar**.

$$\rho_1 = \frac{1}{2} + i14.1386$$
$$\rho_2 = \frac{1}{2} + i$$
$$\rho_3 = \frac{1}{2} + i25.31$$

ii) another formula for the function ζ itself, in fact **Riemann** had for ζ a formula in definite integrals.

The Riemann–Siegel formula

The **Riemann**-**Siegel** formula is a formula that enables us to calculate , to this formula s before to approxiate the zeta function. We begin by

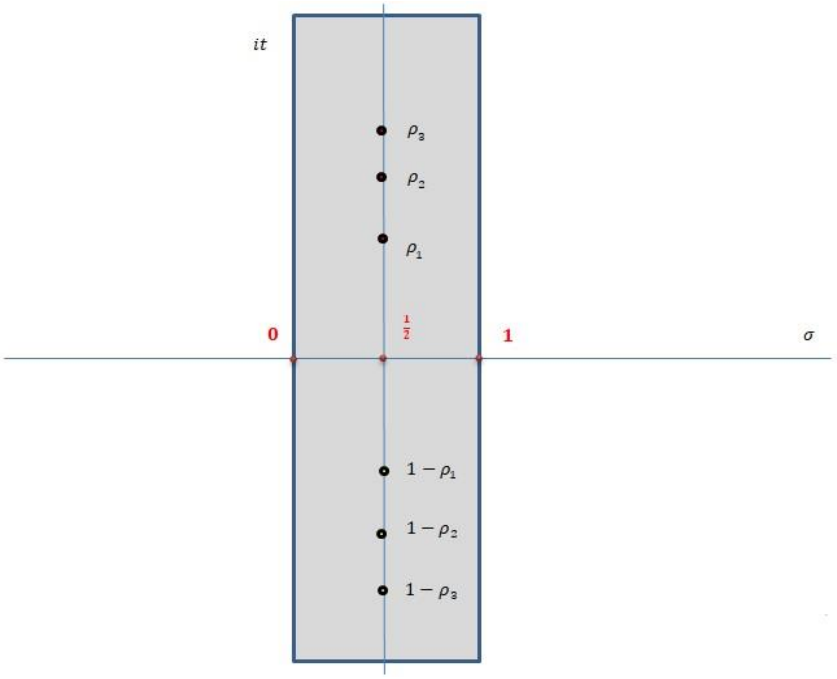


Fig. the three first roots of the zeta function in the critical strip and on the critical line

the aim is to split the zeta function in a more manageable sum by the use of series and definite integrals, so we return to the original integral,

$$\Gamma(z) = n^z \int_0^{+\infty} e^{-nt} t^{z-1} dt$$

So we know that the geometric series $\sum_0 e^{-nx} = \frac{e^x}{e^x - 1}$ to eliminate the indice $n = 1$ which does not exit I the definition of ζ as an infinite series

$$\sum_1^{+\infty} e^{-nx} = \frac{e^x}{e^x - 1} - 1 = \frac{1}{e^x - 1}$$

we finite sum

$$\frac{e^x}{e^x - 1} = \sum_0^N e^{-nx} + \sum_{N+1}^{+\infty} e^{-nx}$$

then

$$\sum_{N+1}^{+\infty} e^{-nx} = \frac{e^x}{e^x - 1} - \frac{1 - e^{-Nx}}{e^x - 1} = 1 + \frac{e^{-Nx}}{e^x - 1}$$

which gives

$$\frac{1}{e^x-1}=\sum_{1}^{N}e^{-nx}+\frac{e^{-Nx}}{e^x-1}$$

so

$$\Gamma(\mathrm{z})(\sum_{n=1}^{N}n^{-z}+\sum_{n=1}^{+\infty}n^{-z})=\int_{0}^{\infty}(\sum_{1}^{N}e^{-nx}+\frac{e^{-Nx}}{e^x-1})x^{z-1}dx$$

When integration

$$\zeta(\mathrm{z})=\frac{e^{-i\pi z}\Gamma(1-\mathrm{z})}{2i\pi}\int_{+\infty}^{+\infty}\frac{x^{z-1}}{e^x-1}dx$$

When we in the integral we find

$$\zeta(\mathrm{z})=\sum_{n=1}^{N}n^{-z}+\frac{e^{-i\pi z}\Gamma(1-\mathrm{z})}{2i\pi}\int_{+\infty}^{+\infty}\frac{e^{-Nx}}{e^x-1}x^{z-1}dx$$

so we have seen that

$$\mathrm{Z(t)}=e^{i\vartheta(t)}\zeta(\frac{1}{2}+it)$$

Becomes after the use of **Siegel** method form **Riemann**

$$Z(t) = 2\sum_{n^2<\frac{t}{2\pi}} n^{-\frac{1}{2}}\cos(\vartheta(t) - t\log(n)) + R_S$$

Where the term R_S is

$$R_S = \frac{e^{-i\vartheta t}e^{-t\pi/2}}{(2\pi)^{1/2}(2\pi)^{it}e^{-i\pi/4}(1-ie^{-\pi i})}\int_{C_N}\frac{(-x)^{-\frac{1}{2}+it}e^{-Nx}}{e^x-1}dx$$

This previous term is the all story and its complex definite integral determined it by using a method **Riemann** invented called the *saddle method* or *steep descent*. It is development of **Laplace** method

We can write

$$\frac{e}{e-1}$$

The new formula for

The path is

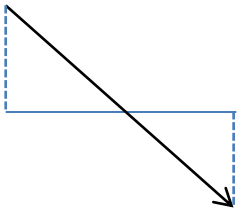


Fig. path of integration

By the method.

$$\int_{C_N} \frac{x^{-\frac{1}{2}+it} e^{-Nx}}{e^x - 2} dx$$

We can write

Lehmer

In the period of 195 and after the **Siegel** discovery and facilitates the calculation of roots of the zeta function, **Titschmarsh** and mathematician **Comrie**(1-19) in the 1935-1936.

In 1953 and beginning of computer programs, mathematician Alan **Turing**(19-195) devised a new method to compute the number of roots in range T and the computer began to write program to calculate and analysis numerical American **Lehmer**(19-19) to calculate in 1955-1956

Rosser law

In the

Exercises

1) let

14

Strategies upon the RH

A lot of research and ideas has stemmed from mathematicians' efforts to understand the **Riemann** zeta function and find a proof to the $\boldsymbol{RH}$, we some those ideas and.

Assertions Equivalent to the Riemann hypothesis:

Through their attempts to prove it, mathematicians have searched for other equivalent assertions to the classical $\boldsymbol{RH}$ in hope that one of them may be less difficult to prove. **Conrey** says

*The **Riemann** zeta function $\zeta(z)$ is the last elementary function that we do not understand. There numerous approaches to $\boldsymbol{RH}$ and hundreds of equivalent formulations in nearly every field of mathematics*[14].

[14] Lately a book in two volume by was published, it contains *equivalence to the Riemann Hypothesis*(2)

i) in 18 French mathematician **Cahen**(18-19) proved that the RH is equivalent to the statement :

$$\sum_{p\leq x} logp \sim x$$

See [**Hadamard** 1896].

ii) Let then we have for $\Re(z) > \frac{1}{2}$

$$\frac{1}{\zeta(z)} = \sum_{1}^{+\infty} \mu(n) n^{-z}$$

The function $\mu(n)$ is called the ***Mobius*** *function*, we have seen before. Then the RH is equivalent to the that $\frac{1}{\zeta(z)}$ has no roots on the $\Re(\frac{1}{2}) > 0$.

iii) let the ***Liouville*** *function* $\lambda(n) = (-1)^{\omega(n)}$, have seen it before $\omega(n) = r$. Then **Landau** proved in his Doctorat thesis (1899) que pour tout $\varepsilon > 0$

$$\boldsymbol{RH} \iff \lim_{n\to+\infty} \frac{\lambda(1)+\lambda(2)+\cdots+\lambda(n)}{x^{\frac{1}{2}+\varepsilon}} = 0$$

iv) En 1901 le mathématicien allemand **Koch** (18-19) montra que

$$RH \Leftrightarrow |\pi(\mathrm{x}) - Li(x)| = Olog^2(x)x^{-1/2}$$

ou de même

$$RH \Leftrightarrow |\pi(\mathrm{x}) - Li(x)| = Ox^{-\frac{1}{2}+\varepsilon}$$

for every $\varepsilon > 0$.

v) Marten made the conjecture if

$$|M(n)| = \left|\sum_{k=1}^{n} \mu(k)\right| < \sqrt{n}$$

then the RH is true. Unfortunately the **Martens** has been proven to be false.

v) if we consider that function seen:

$$\Xi(t) = 2\int_{1}^{+\infty} \Phi(x)cos(2xt)dx$$

Where

$$\Phi(x) = 4\sum_{1}^{+\infty}(2n^4\pi^2e^{9x} - 3n^2\pi e^{5x})e^{-n^2\pi e^{4x}}$$

Then RH is equivalent to the fact that

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Phi(x)\Phi(x)\mathrm{e}^{i(a+b)x}\mathrm{e}^{(a-b)x}(a-b)^2dadb \geq 0$$

vi) This is the *positivity condition* due to André **Weil**[15] namely that the RH is equivalent o the assertion

[15] *Sur "les formules explicites" de la théorie des nombres*(1952). In 2000, **Bombieri** reconsidered this condition and refined it.

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\Phi(x)\Phi(x)e^{i(a+b)x}e^{(a-b)x}(a-b)^2dadb\geq 0$$

vi) ***Lie*** *criteria* (1997) :

$$\boldsymbol{RH}\Leftrightarrow|log\xi(z)|>0$$

due to Chinese Xian-Jin **Lie**(born in 197). In 1999 **Bombieri** and **Lagarias** generalized this criteria to th.

Operator theory

Scientists have suspected since a long time a relation between the zeta function and infinite matrices, and this problematic has seen three directions:

i) the possible existence of an infinite matrix whose eigenvalues are exactly the non-trivial roots of the **Riemann** zeta function(**Landau**, **Polya** , **Hilbert**)

ii) the introduction the random matrices to study the energy of particles resulting in treating three sets of particular matrices : GUE , GOE and GSE ensembles. These preoccupations belong of the theory of probability and statistics used much in quantum mechanics.

iii)**Montgomery** pair conjecture and its developments.

We will here talk about the first and in the next title the two others.

Hilbert Operators

In 19 the Hungarian Goerg **Pólya**(1887-1985) an idea suggested by his teacher **Landau** in 1914, sent a letter to **Hilbert** asserting the possibility that there is a matrix whose eigenvalues are exactly the non-trivial roots of the function ζ . The theory related to matrices and their eigenvalues is called *spectral theory*.

a matrix is a rectangle array of number of variables (functions)

$$\begin{pmatrix} a_{11} & \cdots & a_{11} \\ \vdots & \ddots & \vdots \\ a_{11} & \cdots & a_{11} \end{pmatrix}$$

The most important matrices are square matrices that have the number of and columns because they have number attached them called *eigenvalues* and can be solving the equation

$$det(A - \lambda I) = 0$$

Where $det(A - \lambda I)$ the characteristic polynomial(or function) to any squared matrix. So in the case of the zeta function :

i) a matrix whose eigenvalues exactly the zeros of zeta I this case it is not symmetric.

ii) a symmetric matrix whose eigenvalues are exactly the numbers γ_i in $\frac{1}{2} + i\gamma_i$.

iii)

Very important matrices are *symmetric* matrices having the propriety

$$A^T = \mathrm{A}$$

ans also *orthogonal* matrices having the propriety

$$A^{-1} = A^T$$

in the case I the of A are complex the matrix is **Hermitian**

$$A = \overline{A^T}$$

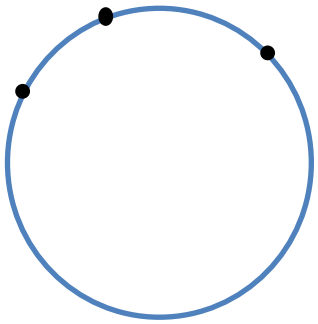
to the French **Hermite**, he proved in 1855 that the eigenvalues of such matrices are all real . it also named *self-adjoint* matrix, the word adjoint here is for transpose. The second is unitary matrix

$$A^{-1} = \overline{A^T}$$

So we have from $\mathbb{R}$ to $\mathbb{C}$

$$\begin{array}{lcl} A\ is\ symmetric & \leftrightarrow & A\ is\ \boldsymbol{Hermitian} \\ A\ is\ \ orthogonal & \leftrightarrow & A\ is\ unitary \end{array}$$

An unitary matrix A has the propriety that its eigenvalues have the module $|\lambda_i| = 1$, so they are on the circum ference of the unit circle



any matrix A is associated to linear map f

$$A \sim f$$

So a matrix is seen as a linear operator, in the same of theory operators in 1918, **Polya** himself (**Edwards** p. 270) a real self-adjoint operator

$$\int_0^{+\infty} f(tx)F(t)dt$$

Has its zeros of transform on $\sigma = \frac{1}{2}$,

Gauss unitary matrix

Quantic mechanic has been formulated as matrix mechanics by German **Heisenberg**(1-197), and the theory have very important. And worked a in terms of matrices as the **Pauli** matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Quantic mechanics is all about *the energy* of the atom and its components like electrons, protons,… in the new mechanic, the electron moves *randomly* in the atom and we cannot predict its position ,momentum,… precisely like in classical mechanics so the physicists thought in terms of probability, the energy level of any system(atoms, particles,…) is described by a law by the eigenvalues of the Hamiltonian operator a Hermitian operator by the Schrödinger PDE

$$i\hbar\Psi_t = \widehat{H}(\Psi)$$

So the probability of at a is by

$$p(s = a) = \|\Psi\|$$

an idea introduced by Max **Born**(1-19).

Random matrices are matrices that have for random variables

$$\begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}$$

They have been introduced in Statistic in 1930s, then in years 1950s, physicist **Wigner**(19-199) found their application related to the energy of the atom particles, when particle or a nucleus get exited the difference between the is seen as the between the eigenvalues of a matrix. since the proprieties of the particles has physically random proprieties, then the matrices is also random.

And a matrix is random is probability if its entries or of random variables like

$$\begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}$$

And it si the normal distribution also known as **Laplace** –**Gauss**. The eigenvalues is distributed as the **Gauss** law or distribution. **Gauss** here is the known density of probability

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\left(\frac{-(x-m)^2}{2\sigma^2}\right)}$$

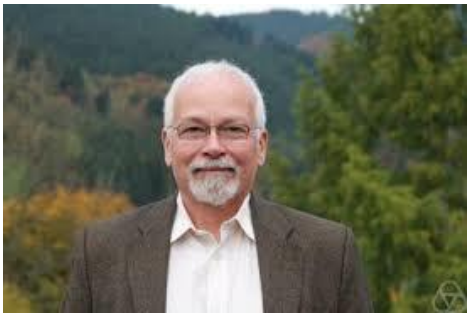
the set of matrices form a group and the found three named by Freeman **Dyson**(1-20):

GOE : Gaussian orthogonal ensemble.

GUE: Gaussian unitary ensemble.

GSE : Gaussian symplectic ensemble.

Pair correlation conjecture

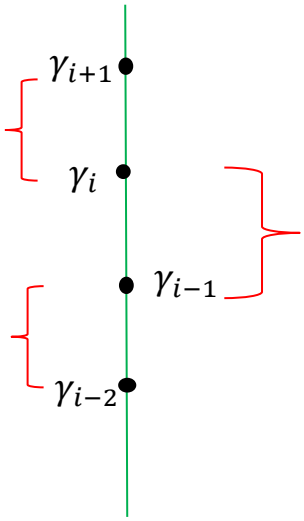


The unexpected discovery of a relation between the spacing between roots of the zeta function and energy levels of particles. In a classic(1971) **Montgomery** studied the difference

$$\left(\frac{1}{2}+i\gamma_{i+1}\right)-\left(\frac{1}{2}+i\gamma_{i}\right)=i(\gamma_{i+1}-\gamma_{i})$$

And to understand the spacing he made some speculation. His conjecture about density of the roots says that :

$$\lim_{T\to+\infty}\frac{number\ of\ (\gamma_{i+1}-\gamma_{i})\propto(a,b)}{N(T)}\ =\ \int_{a}^{b}1-(\frac{sin(\pi x)}{\pi x})^{2}dx$$



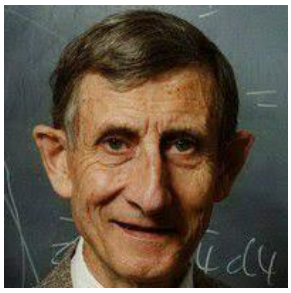


Fig. Spacings between roots on the critical line.

During a historic meeting (1972) between physicist Freeman **Dyson**, a whole new branch of research was born. When **Montgomery** showed his density to **Dyson** he immediately recognized it as the density of eigenvalues of some matrices called random matrices used in quantum mechanics.

The spacing between zeta zeros (i.e. : $\gamma_{i+1} - \gamma_i$) is the same as the spacing between the eigenvalues of random matrices.

Dyson pointed out to **Montgomery** that the density found is the same as the energy levels presented by the matrices of random variables. He explained In a letter to Atle **Selberg**(1-20) dated on April 7th 1992 , that[16]

*The pair-correlation function of zeros of ζ –function is identical with that of eigenvalues of a random (**Hermitian** or unitary) matrix of large order.*

His is thus *pair correlation function* a conception not known to him before the

[16] He indicated to **Montgomery** a reference on the subject : M.L. **Mehta** Random Matrices(1967) p.76 equation 6.13, and p.113 equation 9.61.

meeting and he mentioned it later in the following year (1973) where **Montgomery** wrote[17]:

In a certain standard terminology the Conjecture may be formulated as the assertion that $1-\left(\frac{(\sin \pi u)}{\pi u}\right)^2$ *is the pair correlation function of the zeros of the zeta function. F.J.* ***Dyson*** *has drawn my attention to the fact that the eigenvalues of a random complex Hermitian or unitary matrix of large order have precisely the same pair correlation function. This means that the Conjecture fits well with the view that there is a linear operator not yet discovered.*

It is just like God helps men into research that led directly to Him: as there is a relation between abstractions and reality: the spacing between the zeta roots (abstraction exist only in mind) and the energy level of atoms a reality that we can feel; they are both have the same origin. This means that the interactions in atoms are not chaotic at all, as the spacing between zeros is linked to primes which are not chaotic but presents some pattern at large. Perhaps the $\boldsymbol{RH}$ means there is no randomness no chaos all has a law posed by God.

The function $\frac{sin(x)}{x}$ occurs a lot in physics and it is called the *sinc function*, it is usually noted:

$$a \times \frac{\sin(\pi x a)}{\pi x a} \qquad \text{or} \qquad a \times \boldsymbol{sinc}(\pi x a).$$

It is the **Fourier** transform

$$f(x) = \int f(t) e^{-i2xt} dt$$

of the function

[17] *The pair correlation of zeros of the zeta function* (AMS- 1973).

$$\Pi_a(x) = \begin{cases} 0\,, & -\infty \le x \le -a/2 \\ 1, & -a/2 \le x \le a/2 \\ 0, & a/2 \le x \le +\infty \end{cases}$$

which is known in physics under the names: '$top - hat$', '$rect$' function and '$box\ car$'. When $lima = 0$ then it is the transform of **Dirac**'s δ :

$$\delta(x) = \begin{cases} 0 & if\ x \neq 0 \\ \infty & if\ x = 0 \end{cases}$$

Many works followed the ideas of **Montgomery** to this day and it becomes a new theory by which mathematicians track the classical $\boldsymbol{RH}$. The prediction of **Montgomery** conjecture were conciliated and cheeked by **Odlyzko**(born in 19) known for his work in numerical analysis, he made extensive calculations and found that there is very resemblance between the distribution of nontrivial roots of the zeta function and the distribution of eigenvalues of random **Hermitian** matrices.

In his third edition(2004), **Mehta** himself added a paragraph for **Montgomery** conjecture p. 16:

*One may compare the Riemann zeta function on the critical line to the characteristic function of a random unitary matrix. This can be understood by the fact that the density of the zeros of the zeta function around $N \approx 10^{12}$ is comparable to the eigenvalue density for the **Hermitian** or the unitary matrices of orders 100 or 20.*

*One may compare the **Riemann** zeta function on the critical line to the characteristic function of the random unitary matrix. In other words, consider the variations of the **Riemann** zeta function as one moves along the critical line $\Re(z) = 1/2$ far away from the real axis as well as the variations of the characteristic function of an $N \times N$ random unitary matrix,*

$$f(x) = \zeta\left(\frac{1}{2} + it\right)\,,\ \ t = T + \frac{x}{2\pi} log\left(\frac{T}{2\pi}\right)$$

$$g(x) = det\left[e^{i\theta}.\mathbf{1} - U\right]\ \ ,\ \ \theta = x.\frac{N}{2\pi}$$

Then the limit of large T and N empirically the zeros of $f(x)$ behave as if they were the zeros of $g(x)$; they have the same m-point correlation functions for m=2,3,4,…; they have the same spacing function p(m,s) for m=0,1,…

In the same book there is a discussion of a same possible relation of the other *zeta functions* with physics and random matrices.

Jensen *conjecture*

Jensen made a conjecture made precise by **Polya**: let the polynomial

$$J_d(x) = \sum_{h=0}^{d} c_h \binom{d}{h} x^h$$

be called ***Jensen** polynomial* of degree d. Then **Jensen** conjecture says : the roots of the polynomial associated to the **Taylor** expansion of $z(1-z)\pi^{-\frac{z}{2}}\Gamma(\frac{z}{2})\zeta(z)$ around the point $z = \frac{1}{2}$ are all real. And to give an equivalent to $\boldsymbol{RH}$:

*All **Jensen** polynomials J_d are hyperbolic*

in 2019 , **Griffin** and **Zagier** made some progress in this direction, about which **Bombieri** wrote[18]

*…in this paper the authors introduce a method to study these polynomials which allows the authors to establish hyperbolicity for a large subset of them. …it is a big step forward in making progress along **Jensen**'s line of thought… as a consequence, they are able to show that the Riemann zeta function in the derivative aspect*

[18] PNAS. Vol. 116. June 4 , 2019.

Gaussian Unitary Ensemble(GUE) follows the predicted random matrix model GUE after an appropriate normalization.

Exercises

1) show that

$$|\pi(\mathrm{x}) - Li(x)| = Olog^2(x)x^{-1/2}$$

$\Longrightarrow$

$$|\pi(\mathrm{x}) - Li(x)| = Ox^{-\frac{1}{2}+\varepsilon}$$

1) show that the RH is equivalent to the fact that $\frac{1}{\zeta(z)}$ has no roots on the $\Re(z) > 0$.

1) the set of matrices form a group for $+$. Is it commutative? What about the operation $\times$?

1) say if the transpose of every matrix symmetric and find its inverse

$$\begin{pmatrix}1 & 2\\3 & 1\end{pmatrix} \quad \begin{pmatrix}i & 1\\2 & 3\end{pmatrix} \quad \begin{pmatrix}i & 2\\3 & i\end{pmatrix} \quad \begin{pmatrix}-1 & 2\\3 & 4\end{pmatrix} \quad \begin{pmatrix}3 & 1\\2 & 3\end{pmatrix} \quad \begin{pmatrix}i & 2\\3 & i\end{pmatrix}$$

1) calculate the eigenvalues

$$\begin{pmatrix}1 & 0\\3 & 4\end{pmatrix} \quad \begin{pmatrix}i & 1\\2 & 3\end{pmatrix} \quad \begin{pmatrix}0 & 2\\3 & 4\end{pmatrix} \quad \begin{pmatrix}1 & -1\\3 & 4\end{pmatrix} \quad \begin{pmatrix}i & 1\\2 & 3\end{pmatrix} \quad \begin{pmatrix}i & 2\\3 & 4i\end{pmatrix}$$

1) say if a matrix is unitary

$$\begin{pmatrix}1 & 2\\3 & i\end{pmatrix} \quad \begin{pmatrix}i & 1\\2 & i\end{pmatrix} \quad \begin{pmatrix}i & 2\\3 & 4i\end{pmatrix} \quad (i) \quad \begin{pmatrix}i & 1\\2 & 3\end{pmatrix} \quad \begin{pmatrix}i & 2\\3 & 4i\end{pmatrix}$$

1) calculate the eigenvalues of the following matrices

$$\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix} \quad \begin{pmatrix}2 & 5\\ 7 & 4\end{pmatrix} \quad \begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix} \quad \begin{pmatrix}i & 1\\ 2 & 3\end{pmatrix} \quad \begin{pmatrix}i & 2\\ 3 & 4i\end{pmatrix}$$

1) Show that the transpose of matrices product always has sense.

1) show that : $\det(A^T A) \geq 0$.

1) show that $det(A^{-1}) = \frac{1}{\det(A)}$ (does not exist if $det(A) = 0$).

1) let $AB = 0$, show that if $det(A) \neq 0$ then $B = 0$.

1) find $\begin{bmatrix}1 & 2\\ 4 & 3\end{bmatrix}^{-1}$ by two methods.

1) find the determinant of the matrix $\begin{bmatrix}1 & 2 & 3\\ 1 & 2 & 1\\ 2 & 0 & 1\end{bmatrix}$ by three methods.

1) Show that $A = \begin{Vmatrix}1 & 2\\ 3 & 1\end{Vmatrix}$ verifies its characteristic polynomial.

1) show that if λ is an eigenvalue for A , then λ^{-1} is an eigenvalue for A^{-1}.

1) what is the spectrum of $A = \begin{bmatrix}1 & 1\\ 9 & 1\end{bmatrix}$?

1) show by two methods that $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ positive definite.

1) Is $\begin{bmatrix}2 & 2\\ 4 & -1\end{bmatrix}$ positive definite, semi-positive definite?

1) let $A = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$ show that $A^{-1} = -A$.

1) Show that A^{-1} is unique.

1) show that: $(AB)^{-1} = B^{-1}A^{-1}$

1) $A^2 = A$ (idempotent matrix, idem : *same* potent).show that $\det(A) = 0$ or 1.

1) Show that all idempotent matrices are singular except I.

1) let A idempotent, show that $I - A$ is idempotent. What about $A - I$?

1) show that $\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$ is involutary.

1) Show that **Pauli** matrices : $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, $\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$ and $\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$ are involutory.

1) show that they are **Hermitian**. Are they unitary?

1) Show that $\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}$ is nilpotent. Of what order?

1) if $A^{-1} = A^T$, A is orthogonal $\in O_n(\mathbb{R}) \preccurlyeq GL(\mathbb{R})$. Show that $det(A) = \pm 1$.

1) Show that: $(A+B)^T = A^T + B^T$.

1) Show that: $A + A^T$ is symmetric.

1) let A any matrix, show that AA^T is squared. Show then that AA^T is always symmetric.

1) Show that if A is symmetric then A^{-1} (if it exists) is also symmetric.

1) Let $A = \begin{Vmatrix} 2 & 1 \\ 1 & 4 \end{Vmatrix}$ find its spectrum. What do you remark?

1) Show that every symmetric real is **Hermitian.**

1) Is $A = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$ diagonalizable? What is the *spectral radius* $\rho(A)$? What is its minimal polynomial?

1) A is non-singular iff 0 is not a root of its minimal polynomial.

1) if : $A = P^{-1}BP$, show that $\det(A) = \det(B)$.

1) Show that they are **Hermitian**, and find their spectrum.

1) show that the ζ zeros with negative γ_i verify the pair-correlation conjecture.

1) we define the **Hadamard** product " $\times_H$ " by $c_{ij} = a_{ij} \times b_{ij}$, give the product of the matrices : $\begin{pmatrix} 1 & 4 \\ 0 & 3 \end{pmatrix} \times_H \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$.

1) a **Hadamard** matrix is defined as a squared matrix that contains only 1 and -1 and whose all lines and columns are orthogonal vectors $(v_i * v_j = 0)$. Write **Hadamard** matrices of order 1, 2, 3, and 4. It is believed there exists a **Hadamard** matrix of all orders. (***Hadamard*** *Conjecture)*

1) we define the 2×2 matrix of **Vandermonde**

$$A_V = \begin{pmatrix} \alpha^0 & \alpha^1 \\ \beta^0 & \beta^1 \end{pmatrix}$$

Give the matrix 3×3, and calculate its determinant. We see that $a_{ij} = \alpha^{i-1}$. why does a vendrmonde system has always a unique solution?.

1) A ***Markov*** *matrix* is used in probability, it is a matrix such that the sum of column entries is always 1, give 2×2 Markov matrix.



Purported proofs and disproofs

Since a lot of time people have been trying to solve problems in mathematics. Since the XVIII century given money to solve difficult problems, if one succeeded thus becoming famous and perhaps earn money. But even without becoming known and earning money the solving a problem has its charm joy and for those people. The problems may vary much in difficulty, but sometimes how open problems were solved may very funny , it is in this way that German George **Danzig**(1914-2005) solved two open problems in statistics thinking they were given as a homework by his teacher Jerzy **Neyman**(1894-1981).

Mathematicians and mistakes

Any scientist ,and in our case mathematician, can be of three :

i) makes rare mistakes.

ii) keeps making mistakes.

iii) Publishes already known result.

Making mistakes is not rare for mathematicians. Before the settlement of a problem many people implied make mistakes. And there is no shame in that , the two titans of mathematics **Euler**[19] and **Gauss** made mistakes. A lot of mathematician has made mistaken, beginning from the early attempts of Greek **Proclus** to prove **Euclid** fifth axiom and the quadrature of the circle to our time perhaps volumes to record these mistakes cannot be sufficient.

Some of the mistakes often are good mistakes because the attempts were very decisive for mathematics and science in general because if the problem is not solved at least it founds a new branch of research.

Mathematicians a of his mistake if he is a good or not this is another thing. But how to judge that a mathematician a mistake is really problematic. **Riemann** with his ingenious creativity and who left behind him subjects that kept mathematicians busy since his death, was with his of ***Dirichlet*** *principle*, a point made by **Edwards**

*His "**Dirichlet** principle" is remembered more for the fact that **Weierstrass** pointed out that its proof was inadequate than it is for the fact that it was after all correct and that with it **Riemann** revolutionized the study of **Abelian** functions.*

For the of a new results, this usually by new mathematician in the domain who has not in this part of the theory.

The unique positions of the RH

The **Riemann** hypothesis has a unique position in the history of mathematics for many reasons. One reason is its difficulty; many very difficult problems have been

[19] For **Euler**'s proof of Fermat Last Theorem in the case $n = 3$, see **Edwards** *Fermat's…*p. 45.

solved to, like the **Fermat** Last Theorem and **Poincaré** conjecture. Perhaps one of the signs that the $\boldsymbol{RH}$ is really out of reach is the contradictory existence of persistent proofs and disproves of the hypothesis.

In 2000 Henri **Berliocchi** (born in 1948), a teacher of economics at the university of Paris(France), published a small booklet at *Economica Publishing* , containing the disproof of the $\boldsymbol{RH}$, now this *disproof* is in its third edition (2011)[20] with corrections.

Lately a physicist from India named Kumar **Eswaran** made an announcement of a proof[21] using **Liouville** function $\lambda(n)$ and random walk. On June 20th 2021 , the Indian Committee members that verified the "proof" declared in their report :

*"On the basis of the assessment, this expert committee has concluded that Dr. Kumar **Eswaran**'s proof of the **Riemann** Hypothesis is correct".*

David **Hilbert** , the person who pointed out the $\boldsymbol{RH}$ as a great problem challenging future mathematics in 1900 ,has even underestimated the problem . He once mentioned three of his 23 problems : the transcendence of $2^{\sqrt{2}}$, **Fermat** Last Theorem and $\boldsymbol{RH}$. He declared that the $\boldsymbol{RH}$ would probably be solved soon, the **Fermat** conjecture in his lifetime, but the transcendence of $2^{\sqrt{2}}$ possibly never[22]. But then he changed his mind

*If I would be awaken after five hundred years I would ask: has the **Riemann** hypothesis been solved!*

Siegel asserted that **Riemann** did not make any step to solve his hypothesis in his Nachlass, , and **Riemann** himself in his paper has made it clear by asserting his "futile attempts" to prove it.
Attempts of non-known mathematicians are usually without success, another problem is that when a person claims a proof it is not easy to find who will verify it. Usually, professional mathematicians refuse to waste time for such a task.

[20] Published also at *Omniscriptum Publishing* in 2016.
[21] *It seems that it is not a proof at all, only some heuristic observations in favor of the* $\boldsymbol{RH}$.
[22] What happened was really the reverse!! We now are waiting only for the $\boldsymbol{RH}$ solution.

Stietjes

The first man in the story about alleged proofs was the known Dutch mathematician **Stieltjes**(18-189). **Martens** made the conjecture

$$|M(n)| = \left|\sum_{k=1}^{n} \mu(k)\right| < \sqrt{n}$$

and this implies the $\boldsymbol{RH}$**.** In fact the $\boldsymbol{RH}$ is equivalent for any ε to

$$M(n) = O(x^{\frac{1}{2}+\varepsilon})$$

So in 1885 **Stieltjes** published a note that he had a proof of a new estimation, that there is a constant C such that

$$\frac{M(n)}{\sqrt{n}} < C$$

The conjecture of **Martens** has been disproved in 1985 by **Odlyzko** and **te Riele**. Although the falsity of **Martens** conjecture does not disprove **Stieltjes** claim, the proof has never been found.

Rademacher :

There is another anecdote about **Hilbert** with $\boldsymbol{RH}$, it says that one of his students in Gottingen presented him with a false proof of the **Riemann** hypothesis and that afterwards the student unfortunately died:

Hilbert *asked the grieving parents if he might be permitted to make a funeral oration. While the student's relatives and friends were weeping beside the grave in the rain,* ***Hilbert*** *came forward. He began by saying that a tragedy it was that such a gifted young man had died before he had had an opportunity to show what he could accomplish. But, in spite of the fact*

that this young's man proof of the RH contained an error, it was still possible that someday a proof of the famous problem would be obtained along the lines which the deceased had indicated. In fact, let us consider a function of a complex variable[23]*…*

This time we mention not a proof, but a disproof by a renowned mathematician who made good work in number theory and has written a books on analytic number. In 1925 German Hans **Rademacher**(1-19) claimed that the if $\boldsymbol{RH}$ is true then the series

$$\sum_{\gamma}\frac{1}{\gamma}sin(\gamma logx)$$

has jumps discontinuities, where γ is the imaginary parts of ζ roots. In 1966 it has been proved that some similar series had such discontinuous even if the $\boldsymbol{RH}$ is false.

It seems that by the same idea **Rademacher** claimed a disproof the $\boldsymbol{RH}$ by contraposition in 1945 . the disproof was found wrong by **Siegel**. The problem is that **Rademacher** had a mistake with the logarithm of a function of complex variable,

DeBrange:

The last one in our who made some noise is **Debrange**(born in 19) he is of French descent who made his carrier as a mathematician in the united States in and he is known to his fellow mathematicians for solving the **Bieberbach**[24] conjecture in 1985, concerning a certain complex power series function

$$\sum a_n z^n$$

where he that

$$|a_n| \leq n|a_1|$$

[23] Reid : **Hilbert**(19) , via Prime Obsession.
[24] A conjecture made in 1916 by German Ludwig **Bieberbach**(1886-1982).

the proof used the **Riemann** mapping theorem. his hope to prove the RH was old , when he was just in his , and he by book "***Hilbert*** *space for Entire Function*" (1968) and his approach is to consider the zeta function as a function in a **Hilbert** space, and work with operator between two spaces and transform the zeros. his effort culminated in a proof *paper entitled "the **Riemann** hypothesis for **Hilbert** spaces of entire functions"*(1986) which is turned to be false. The **Debrange** tools used to attack the $\boldsymbol{RH}$ consist in function spaces, and considering the function $\xi(z) = 2\Gamma(z)\zeta(z)$ as a function that has some proprieties, it is a **Hardy** space $\mathcal{F}(W)$ of Analytic functions. In summer of 1989 he presented in Paris another false proof.

In 2004 he announced that he solved by using **Hilbert** space and to focus on gamma function. Mathematicians community were very skeptic, and some working on the problem like B. **Conrey** and P. **Sarnack** pointed out that his method contradict by.

It is thought that the problem has nothing to do with function spaces. In his paper, **Conrey** stated:

> *In my belief that $\boldsymbol{RH}$ is a genuinely arithmetic question that likely will not succumb to methods of analysis. There is a growing body of evidence indicating that one needs to consider families of L-functions in order to make progress on this difficult question. If so, then number theorists are on the right track to an eventual proof of $\boldsymbol{RH}$, but we are still lacking many of the tools. The ingredients for a proof of $\boldsymbol{RH}$ may well be moment theorems for a new family of L-functions not yet explored; modularity of **Hasse-Weil** L-functions for many varieties, like that proved by **Wiles** and others for elliptic curves; and new estimates for exponential sums, which could come out of arithmetic geometry.*

Debranges published in 2010 a paper of 43 pages under the title *"Apology for the proof of the **Riemann** hypothesis*" where he explains his failure. And on April 13th 2017 , he published another proof. So number theorists are now convinced that his approach will not prove the hypothesis.

One of **Debranges** students, Xing **Lie** , known for the ***Lie*** *criterion*, announced a proof in June 2008 which he began by saying:

*By using **Fourier** analysis on number fields, we prove in this paper E. **Bombieri**'s refinement of A. **Weil**'s positivity condition, which implies the Riemann hypothesis for the Riemann zeta function in the spirit of A. **Connes**' approach to the **Riemann** hypothesis.*

Li founded his proof on the positivity condition that has been established by **Weil** in two papers : one in 1952 and the other in 197. We Remind of the **Riemann** explicit formula

$$\Pi(x) = Li(x) - \sum_{Im\rho \geq 0}[Li(x^{\rho}) + Li(x^{1-\rho})] + \int_{x}^{\infty} \frac{dt}{t(t^2-1)logt} + log\xi(0)$$

So if $\rho = \frac{1}{2} + i\gamma$, **Weil** generalized it to

$$\sum_{\gamma} h(\gamma) = 2h(\quad) - g(0)log + \int h(r)dr - 2\sum_{n\geq 1} \frac{\Lambda(n)}{\sqrt{n}} g(logn)$$

where is the von **Mangoldt** function and he gave the positivity condition. A paper of **Connes**(1999) and **Bombieri**(2000)

$$\int_{k_v^*} \frac{h(u)}{|1-u|} dx$$

Bombieri it as the **Mellin** transform involving the zeta zeros ρ

$$\sum_{\rho} \int_{0}^{+\infty} h(x)x^{\rho-1}dx > 0$$

in his paper, **Li** defined

$$\Delta(h_0) = h_0 + h_0 + \sum_{\rho} \int_{0} \frac{h(x)x^{\rho-1}dx}{|\quad|} dx$$

he affirmed that if

$$\Delta(h_0) \geq 0$$

then the $\boldsymbol{RH}$ follows. But a flaw has been spotted by **Connes** in that his **Fourier** transform on adels of measure zero. the paper withdrew by the author.

Exercises

Find the inconstancy or mistake in every case:

1) if we begin by $x = 0.999\ldots$ then multiplying by 10 : $10x = 9.999\ldots$, subtracting $(10x - x) = (9.999\ldots - 0.999\ldots)$ we find : $x = 1$.

1)

$$-1 = i \times i = \sqrt{-1} \times \sqrt{-1} = \sqrt{-1 \times -1} = \sqrt{1} = 1$$

1)

$$0 = log(1) = log(-1)(-1) = log(-1) + log(-1) = 2\pi i \ ?$$

1) Here is a proof of the **Riemann** hypothesis :

The function ξ has the same roots as ζ in the critical strip and they are 4 zeros for the same $|t|$. As $\xi(z) = \xi(1 - z) = 0$ both z and $1 - z$ are zeros, and the two others they come from $\overline{\xi(z)} = \xi(\bar{z}) = 0$ and $\overline{\xi(1 - z)} = \xi(\overline{1 - z}) = 0$. So we have

$$z = \overline{1 - z} \qquad \wedge \qquad \bar{z} = 1 - z$$

In the two cases we can write

$$\sigma_0 + it = \overline{1 - (\sigma_0 + \imath t)} = 1 - \sigma_0 + it$$

which gives : $\sigma_0 = \frac{1}{2}$ ■

1) which estimation of $M(n)$ is stronger, **Martens**' or **Stieltjes**' ?

The *Zeta* functions

After **Riemann** paper, mathematicians have been interested in number theory and its relationship with analysis, they have invented other *zetas functions* of the form

$$\zeta_K(z) = \sum_{1}^{+\infty} \frac{a_n}{n^z} = \prod^{+\infty} (1 - \frac{a_n}{p^z})^{-1}$$

with new proprieties according to the values of the coefficients a_n in the main term $\frac{1}{n^z}$. The nature of these numbers is algebraic and linked to *modulo n* , thus a new theory called *algebraic number theory*, a link between algebra , analysis and number

theory , it is really one of the crowning achievements of human mind of the XX century.

The **Riemann** zeta function itself is a special case of a general set of functions called **Dirichlet** series

$$\sum_{1}^{+\infty} \frac{a_n}{n^z}$$

a are In **Ivic** [$\boldsymbol{p}$.v]

*…Another important aspect of the theory of **Riemann** zeta-function that is that $\zeta(s)$ the simplest of a large class of the **Dirichlet** series known as zeta-functions. General zeta-functions occur in many branches of mathematics, including algebraic and analytic number theory, and results from the theory of $\zeta(s)$ in many instances generalize to the zeta-functions.*

Dirichlet

The work of **Dirichlet** preceded **Riemann** zeta function , on two subjects

$*$) Primes in arithmetic progressions.

$**$) Binary quadratic forms.

and he created functions of the form

$$L_\chi(z,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^z} = \prod_p (1 - \frac{\chi(p)}{p^z})^{-1}$$

The $\zeta(z)$ function is only a particular case of L_χ where $\chi(n) = 1, \forall n \in \mathbb{N}$.

Having different proprieties from the infinite series $\sum_{n=1}^{\infty}\frac{1}{n^z}$, the infinite series $\sum_{n=1}^{\infty}\frac{\chi(n)}{n^z}$ converge when : $\sigma > 0$. And there very between the two series

$$\sum_{n=1}^{\infty}\frac{\chi(n)}{n^z} \qquad and \qquad \sum_{n=1}^{\infty}\chi(n)e^{-nz}$$

As **Weil** [25] noted , the Swedish mathematician Carl **Malmstén**(1814—1886) published a full proof in 1846. Later in 1849 German Oskar **Schlömilch** (1823-1901) proved the functional equation for the L_χ function in the range : $0 < \sigma < 1$. Also German Gotthold **Eisenstein** (1823-1852) proved the functional equation of L —function modulo 4.

$$L_\chi(z,\chi) = \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)^z} = 1 - 3^{-z} + 5^{-z} - 7^{-z} + \cdots$$

see also **Cahen**(1894).

Dedekind

In (189), **Dedekind** developed a general algebraic zeta function

$$\zeta_K(z) = \sum_{A}|A|^{-z} = \prod_{P}\frac{1}{1-|P|^{-z}}$$

Where in place of the prime number p, P are the prime ideals of the *number field* $\mathbb{K}$,. when the number field is $\mathbb{K} = \mathbb{Q}$, we find the **Riemann** zeta function.

is a new that number field a number field is where the pole 1 is rsidu

[25] *On Eisenstein's copy of the Disquisitiones* (ASPM, 1989, p.463-469)

$$= \frac{\Delta}{}$$

And he

Hurwitz

German mathematician **Hurwitz** (1-19) developed a zeta function depending on a real number a:

$$\zeta(z,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^z}$$

with $0 < a \leq 1$. Unlike the preceded, this function can have an **Euler** product in prime numbers only for $a = 1$, in this case it's the **Riemann** zeta function, or when $a = \frac{1}{2}$

$$\zeta\left(z,\frac{1}{2}\right) = \sum_{n=0}^{\infty} \frac{1}{(n+\frac{1}{2})^z} = 2^z \sum_{n=0}^{\infty} \frac{1}{(2n+1)^z}$$

And we know

$$\zeta(z) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^z} + \frac{1}{2^z} \sum_{n=1}^{\infty} \frac{1}{n^z}$$

which gives

$$\zeta\left(z,\frac{1}{2}\right) = (2^z - 1)\zeta(z)$$

Also we have

$$L(\chi,z)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{z}}=k^{-z}\sum_{n=1}^{k}\chi(r)\zeta\left(z,\frac{r}{k}\right)$$

See **Apostol**(p.249) for a proof.

Hecke

The Dedekind zeta function were developed further by German mathematician **Hecke** (1-19) , **Dedekind** conjectured in 1877 that

$$\zeta_K(z)(z-1)$$

may be continued to an entire function for any $z\in\mathbb{C}$, a result proved by **Hecke** in 1917 for every algebraic number field.
Hecke has also prove for the **Dedekind** function

$$=\sum$$

a la fonction zêta des .

Epstein

German Paul **Epstein**(1871-1939) developed in 1903 another zeta function defined as

$$\mathcal{Z}_{a,b}(z)=\sum_{n=-\infty}^{n=+\infty}\frac{e^{2\pi inb}}{(a+n)^{z}}$$

Where a and b are real numbers, the idea came from **Riemann**'s second form of the zeta function using of theta function $\varphi(t,x)=\sum_{n=-\infty}^{n=+\infty}e^{-(n+x)^2\pi t}$, so he found

$$\pi^{\frac{-z}{2}}\Gamma(\frac{z}{2})\mathcal{Z}_{a,b}(z)=\int_0^{+\infty}\varphi_{a,b}(0,x)x^{\frac{z}{2}-1}dx$$

And the functional equation

$$e^{2\pi iab}\pi^{\frac{-z}{2}}\Gamma\left(\frac{z}{2}\right)\mathcal{Z}_{a,b}(z)=\pi^{\frac{z-1}{2}}\Gamma\left(\frac{1-z}{2}\right)\mathcal{Z}_{a,b}(1-z)$$

Artin breakthrough

In his PhD thesis of 1921, German mathematician Martin **Artin** (19-19) introduced some new ideas in number theory, his aim was to follow **Dedekind**'s classical studies on number fields as usual, like class number, class number formulas... In this time the number field of **Dedekind** is replaced a new field called *function field* [26] whose elements are functions. These functions are considered as curves whose coefficients are in a finite filed $\mathbb{F}$ and they are characterized by geometry proprieties like their genus g.

Artin worked in quadratic $\mathbb{F}\left(x,\sqrt{D}\right)$ of hyper-elliptic function fields, as with **Dedekind**

$$\zeta_K(z)=\sum_A|A|^{-z}=\prod_P\frac{1}{1-|P|^{-z}}$$

Where P are the primes of K and A are the *integer divisors* of K. It is also written

[26] Noted also noted or $F|\mathbb{F}$ where $\mathbb{F}$ is the base field.

$$\zeta_V = e^{\sum_{n\geq 1}\frac{N_n}{n}T^n} = exp\sum_{n\geq 1}\frac{N_n}{n}T^n$$

In fact **Artin** did a big work by making a transition of number theory from number fields in the spirit of **Dedekind,** to new fields which are function fields[27]. As an example of the analogy between numbers and the polynomials the result that there infinitely many primes in $\mathbb{Z}$ is analogous to the infinitude of the number of monic polynomials $(x + a)$ in $\mathbb{F}[x]$ and the propriety in $\mathbb{Z}$

$$\sum_{p\geq 2}\frac{1}{p} = +\infty$$

Is similar in

$$\sum_{p}\frac{1}{q^{deg\, p(x)}} = +\infty$$

where $p(x)$, are the irreducible polynomials. This comes from the fact that the number of irreducible polynomials when their coefficients are in $\mathbb{F}$, is the number $q^{deg\, p(x)}$.

Artin defined zeta function for the new function fields over a finite field

$$\zeta_K(z) = \frac{L(t)}{1 - qt}$$

Where $L(t)$ is a polynomial, anyone wants to understand this first o understand the work of Dedekind on zeta of algebraic number fields.

and were that the **Riemann** hypothesis was true for many numerical examples. The roots are

[27] See: *the **Riemann** characteristic its history*(2018) by.

$$\frac{1}{q} < |t| < 1$$

He gave a criterion for a similar classical $\boldsymbol{RH}$ for this new fields as

$$\left|N_{R_r} - q^r\right| < A\sqrt{q^r}$$

where R_r is the number of prime ideals in the ring R.

The new **Riemann** hypothesis become $\boldsymbol{RH_p}$: the **Riemann** hypothesis of characteristic p. The classical $\boldsymbol{RH}$ belong to number fields which have characteristic 0.

First to prove similar proprieties of the **Riemann** zeta function in the case of function fields, a lot of work had to be done so

i) Rationality of ζ_V.

ii)Equivalent of the $\boldsymbol{RH}$ and its proof.

First German F.K. **Schmidt** (1901-1976) gave the definition of the zeta function for any function field over a finite field not only those studied by **Artin**. Second he established the functional equation of ζ in function field by proving the **Riemann-Roch** theorem for function fields namely an equivalent to

$$l() = g - n + 1$$

that it is equivalent to for function field .

Afterwards another German mathematician Helmut **Hasse** (1898-1979) studied with **Davenport**(1-19) the problem of estimating the number $\#$ of integer solutions of an equation where the coefficients and the solutions are in a finite field $\mathbb{F}\ q$ of elements[28] . **Artin** discovered that this problem is equivalent to the **Riemann** hypothesis for function fields namely $\boldsymbol{RH_p}$: .

[28] Any student of algebra know that the cardinal of a finite field $\mathbb{F}$ is of the form $q = p^n$ where p is a prime number.

Hasse managed in 1934 to prove the $\boldsymbol{RH_p}$ for elliptic function field containing polynomials of the form

$$y^3 = x^2 + x + c$$

And then in he for general **Fermat** filed fields, i.e. fields of functions

$$ax^m + by^n + c = 0$$

a, b and $c \in \mathbb{F}$.

Weil conjectures



Hearing of the developments, French mathematician André **Weil**(1908-1998) , with **Hasse** has an idea about generalizing the ideas of **Riemann** and to general curves and he first And constructing it a **Riemann** hypothesis, and he succeed in proving it in the period 1936-38. a **Riemann** hypothesis for and succeed to prove it in 19--41-1948.

Weil result

The $\boldsymbol{RH_p}$ is true for general function fields.

Then in 1946 **Weil** published his influent book on new algebraic geometry following Italian geometers(**Seviri**,..) but in characteristic p, as classical geometry was in $\mathbb{C}$ a field of cjarectrestic 0.

fonction ζ associated an algebraic variety

$$\sum \frac{a}{n}$$

Algebraic variety is simply a generalization of the concept of curves and surfaces.

le cadre de la géométrie algébrique these arethe famous: *Weil conjectures* .

i) Rationallity of ζ_V.

ii)Equivalent of the $\boldsymbol{RH}$ and its proof.

And the third :



after a lot works done in the 1950' especially by French algebraists as **Serre**, **Grothiendiek**, who developed new ideas to attack the conjectures:

- it has been proved results the functional equation and

-

- it has been proved results the functional equation and

the last one the **Riemann** hypothesis proved by other methods by Belgium Pierre **Deligne** in 1974. **Deligne** did i no use ideas form topology as the .

there also a general hypothesis for all L-function which qui les mathematicians believe is true:

Generalized **Riemann** hypotheses (GRH)

All the non trivial zeros of any zeta function have the reat part $\sigma = \frac{1}{2}$

the classical **Riemann** hypotheses still resists to all known technics, and sadly all these result remains feeble. **Conrey**(2003) says:

The study of L-functions is still in its beginning stages. We only recently learned the modularity of the L-functions associated to elliptic curves; it would be very helpful to understand the L-functions for more complicated curves and generally for varieties. It would be useful to systemically compute many new examples of L-functions to get a glimpse of what is out there waiting to be discovered. The exotic behavior of the multiple zeros of L-functions associated to elliptic curves with many rational points could be just the beginning of the story.

Exercises

1) show that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)^z} = 1 - 3^{-z} + 5^{-z} - 7^{-z} + \cdots$$

converges absolutely for $\sigma > 0$.

1) show that **Hurwitz** function $\zeta(z,a) = \zeta(z)$ for $a = 1$.

1) in the **Epstein** zeta function show that

$$\pi^{\frac{-z}{2}}\Gamma(\frac{Z}{2})\mathcal{Z}_{a,b}(z) = \int_0^{+\infty} \left(\sum_{n=-\infty}^{n=+\infty} e^{-\pi x(a+b)^2 + 2\pi i n b} \right) . x^{\frac{z}{2}-1} dx$$

1) show that

$$\mathcal{Z}_{0,b}(z) = 2\sum_{n=1}^{n=+\infty}\frac{\cos(2\pi nb)}{n^z}$$

Entire functions

The search for a proof of the prime number theorem an the Riemann hypothesis after **Riemann** has developed much two branches:

i) Complex analysis

ii) Research on the entire functions and the distribution of their roots. They are functions that are finite and analytic in any finite domain.

Entire function looks like polynomial , they have no singularities, then their order of growth is good. Polynomials are *entire rational function* and the other function that are not polynomial and they are entire are called: *entire transcendental functions* like, ...

-growth of an entire function is linked to its zeros

-it is not necessary that f be an entire to be representable a product as its zeta , gamma sin.

Riemann chercha to write the function ξ by its non-trivial roots, the same as we write the expression of a polynomial its roots, an idea may be to one is to construct a function as a product of infinitly many polynomial with zeros that has $\sigma = \frac{1}{2}$, in fact the re result by **Polya** in 1927 (**Edwards** p270) that if $p(x)$ is a polynomial with only pure imaginary roots, if

$$\int_0^{+\infty} t^{-z} F(t) dt$$

Has its zeros on $= \frac{1}{2}$, then so has the function

$$\int_C F(t) p(log t)\, dt$$

Where F is a function satisfying $t^{-1}F(t^{-1}) = F(t)$.

To do his task **Riemann** , eliminated first the trivial roots but the question is

The zeros comes from the fact that

$$\frac{1}{2\pi i}\int_C \frac{\xi'(z)}{\xi(z)} dz \neq 0$$

For $\sigma > 1$ there a correspondence between the non-trivial root of the function η roots and the primes number through the heuristic equation

$$\eta(z) = f(z)(1 - 2^{1-z})^{-1} \prod_\rho \left(\frac{z}{z-\rho}\right)^{-1} = (1 - 2^{1-z}) \prod_p \left(1 - \frac{1}{p^z}\right)^{-1}$$

And the other zeros of are regular unlike the prime numbers, so

$$\xi(z) = \xi(0)\prod_{\rho}(1-\frac{z}{\rho})$$

if we designate by α every root of the equation $\xi(\alpha) = 0$, we then can exprimer log $\xi(t)$ by

$$\sum log\left(1-\frac{t^2}{\alpha^2}\right) + \log\xi(0)$$

Riemann by writing g this he suppose we may write the function as a product la fonction

$$\xi(t) = \xi(0)\prod_{\gamma}(1-\frac{t^2}{{\gamma_i}^2})$$

where γ_i are zeros of $\xi\left(\frac{1}{2}+it\right)$ and thus of . taking the logarithm of the two members we find his result , he considered ξ comme un polynomial of infinite degree .

Riemann from the real case let the polynomial $p = x^2 + x - 6$ then

$$p = (x-2)(x+3)$$

When we find xx. The number is the product of the roots 2 and -3. then we can

write

$$p = (x-)(x-)$$

in case is not 1 like $p = 2x + x + 3$ we have

$$p = (x-)(x-)$$

if $p_n(x)$ is a polynomial of degree n on peut l'écrire comme un produit de ses zéros $x_1, \dots, x_n$

$$p_n(x) = p_n(0) \prod_{zeros} (x - x_i)$$

Or the same

$$p_n(x) = p_n(0) \prod_{zeros} x(1 - \frac{x_i}{x})$$

If $p_n(0) = 0$ then

$$p_n(x) = x \prod_{zeros} (x - x_i)$$

This for

$$x^4 - 1 = (x + 1)(x - 1)(x^2 + 1)$$

but when a function has infinitely many zeros it is not evident. We have noted that the product is linked to the distribution of the zeros and also to the $rate\ of\ variation$ of f. A problem: has two roots ρ and $1 - \rho$ so which is convenient

$$\prod_{zeros}(1-\frac{z}{\rho})(1-\frac{z}{1-\rho})$$

or

$$\prod_{zeros}(1-\frac{z}{\rho})\prod_{zeros}\left(1-\frac{z}{1-\rho}\right) \quad ?$$

Hadamard showed that the first product should be the .This form existed already in the writing of **Euler** who proved that

$$sin(\pi z) = \pi x\prod_{n}(1-\frac{x^2}{n^2})$$

Ana analytic function is seen as an infinite polynomial as a power series

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$$

so i t is naturel to the its zeros .

this especially later by **Weierstrass** in 1876 consider, une fonction entière ressemble à un polynôme car il n'a pas de singularités. The products of **Weierstrass**(18) qui les factors primaires

$$E(z) = 1 - \frac{1}{p^x}$$

Est defined as follows :

Weierstrass 1876

Let f then

$$f(z) = z^m e^{g(z)} \prod_p (1 - \frac{1}{p^x})$$

Et il a écrit un produit pour la fonction Gamma. Une théorie de ces fonctions a été développée plus à la suite des travaux sur la fonction zeta. L'idée est qu'une fonction entière est classée selon son *ordre de croissance* on dit qu'une fonction entière est ordre n si

Maximum module principle

the result of **Borel-Caratheodory**.

esults about these particular functions

maximum theorem du (18)

if a function f est entire admet un maximum locale alors elle est constant

Ce qui veut dire qu'une function ne peut avoir de maximum local. Et aussi le théorème du à **Cauchy** (1844) , but known as *theorem of Liouville.*

Liouville theorem :

Every bounded entire function f is a constant.

in fact **Liouville** proved it for a case, it was one of his students that the name. and the more general le grand theorem due to Emile **Picard** we have seen for meromorphic functions:

Picard little theorem(1880) :

Every entire function having can take infinitely many times except at most one.

s an example the function f(z)= . Then

***Picard** big theorem(1879) :*

Every non-constant holomorphic function f can take all the finite complex values in $\mathbb{C}$ except perhaps one value at most.

This different from his little theorem is the essential singularities like the example: $f(z) = e^z$ which does not take the value 0.

Also tre from entire order and zeros:

Theorem (1880) :

every entire function of order 1 has infinitely many zeros.

This theorem in **Titshmrsh**(I) p. 15 where to **Valiron**. **Hadamard**(1893) en s'appuyant on the result of **Weierstrass** des fonctions entières.

$$f(z) = \prod_p (1 - \frac{1}{p^x})^{-1}$$

Riemann que la fonction peut la fonction ξ à l'aide de ces zéros réels, en admettant que l'hypothèse est vraie les zéros sont de la forme $\frac{1}{2} + i\alpha$

$$\xi(t) = \xi(0)\prod_{\alpha}\left(1-\frac{\frac{1}{2}+it}{\frac{1}{2}+i\alpha}\right) = \xi(t) = \xi(0)\prod_{\alpha}\left(\frac{\frac{1}{2}+it}{\frac{1}{2}+i\alpha}\right)$$

$$= \xi(0)\prod_{\alpha}\left(1-\frac{\frac{1}{2}+it}{\frac{1}{2}+i\alpha}\right)$$

$$= \xi(0)\prod_{\alpha}\left(1-\frac{\frac{1}{2}+it}{\frac{1}{2}+i\alpha}\right)$$

$$= \xi(\frac{1}{2})\prod_{\alpha}\left(1-\frac{\frac{1}{2}+it}{\frac{1}{2}+i\alpha}\right)$$

Car =
c'est la forme donnée par **Riemann**. Si on la de hypothèse de **Riemann** on a les zéros sont ρ et $1-\rho$

$$\xi(z) = \xi(0)\prod_{\alpha}\left(1-\frac{z}{\rho}\right)\left(1-\frac{z}{1-\rho}\right)$$

we take the naperian logarithm we find

$$log\xi(z) = log\xi(0) + \sum log\left(1-\frac{z}{\rho}\right) + log\left(1-\frac{z}{1-\rho}\right)$$

comme trouvé par **Riemann**

donne de son raisonnement en disant

$$\left(1-\frac{z}{\rho}\right)\left(1-\frac{z}{1-\rho}\right) = \left(1-\frac{z(1-z)}{\rho(1-\rho)}\right)$$

Hadamard theorem

Hadamard that the work of weiertrass has genre of a function in the product

$$f(z) = e^{G(z)} \prod_p (1 - \frac{x}{\xi_p}) e^{Q_p(x)}$$

the concept:

Definition

we say that a function f is of the genre E , of if the degree of the polynomial $Q_p(x)$ are of degree n and the function $G(x)$ at most n.

and **Poincaré** he to that the coefficient if . **Hadamard** assets:

*"The analysis of **Riemann** repose on the fact that $\xi(x)$, considered as a function of x^2, is of genre 0, but without a sufficient proof. this researches will us to determinate with rigor the genre of ξ ".*

Then he result about zeta function

Hadamard's theorem (1893)

For all z

$$\xi(z) = -e^{-Az} \prod_\alpha \left(1 - \frac{z}{\rho}\right) e^{\frac{z}{\rho}}$$

where are the roots and

$$A = -\frac{\gamma}{2} - 1 + \frac{1}{2} log(4\pi)$$

the paper of **Hadamard** and its proof is very general and to function more than, but pour pas pour mais pour une large partie des fonctions entières. On a aussi

$$\zeta(z) = \frac{1}{\prod_{\alpha}\left(1 - \frac{z}{\rho}\right)}$$

Theorem

if is of order 1 then it has infinitely many zeros

The problem of the roots

since the by roots theri density or fequenct in the criticla strip.

In fact, since the density of the roots of size t grows with t only as does $log(\frac{t}{2\pi})$ ***, this expression converges, and for t infinite becomes infinite only as*** $tlogt$***;***

Riemann syas at the end of his article and in his collected works

Is has not proved

Pour cela Riemann a dû d'abord étudier comment les zéros de ζ are distribued in the critical strip, soit un zéro de alors on a une information sur t, il dit that

$$tlogt$$

And they are fairly dense, we can say approximately there is always a root between consecutive integers n and $n+1$.

Theorem:

If $\sum|a_n| < +\infty$ alors $\prod_n(1+a_n) < +\infty$

elle diffère de $log\xi(t)$ par conséquent d'une fonction de t^2 qui, pour t fini, reste finie et continue et qui, divisée par t^2 , sera infiniment petite pour t infini.
Cette différence, par suite, est une constante dont la valeur peut être déterminée en posant $t = 0$.

the product esquissed by **Riemann**

$$\xi(z) = a\prod_{\rho}(1-\frac{z}{\rho(1-\rho)})$$

he means take log

$$log\xi(z) = loga + log\sum_{\rho}(1-\frac{z}{\rho(1-\rho)})$$

or as he wrote for $log\xi(t)$

$$\sum log\left(1-\frac{tt}{\alpha\alpha}\right)+log\,\xi(0)$$

then

$$\log\xi(z)-log\sum_{\rho}(1-\frac{z}{\rho(1-\rho)})=loga$$

thus he means the difference $log\xi(z)-log\sum_{\rho}(1-\frac{z}{\rho(1-\rho)})$ must be a constant as he expect and suppose, by saying that

$$\log\xi(z)-log\sum_{\rho}(1-\frac{z}{\rho(1-\rho)})\sim zlogz$$

was proved by **Hadamard** in 1893.
Aussi à l'aide du théorème de **Hadamard**, la fonction ζpeut-être écrite à l'aide se non trivial roots

$$\zeta(z)=\frac{e^{\mathrm{z}(\log(2\pi)-1-\frac{\gamma}{2})}}{2(\mathrm{z}-1)\Gamma(1+\mathrm{z}/2)}\prod_{\rho}(1-\frac{z}{\rho})\,e^{z/\rho}$$

We notice that $z=1$ is a singularity.

Exercises

1) write teh folowing polynomials as : $p_n(x) = p_n(0) \prod_{zeros}(x - x_i)$

i) $p_1(x) = 2x^2 - 3x$

ii) $p_2(x) = 2 - 3x - 2x^2$

iii) $p_3(x) = 2 - 2x^2 + 3x$

i) $p_1(x) = 2 - 3x$

ii) $p_2(x) = 2 - 3x$

iii) $p_3(x) = 2 - 3x$

1) let p a polynomial show that if $p(x) = (x - a) \times q + b$, then $b = f(a)$.

2) trouver le rayon de convergence des séries

$\sum_1 \frac{z^n}{n^2(1-z^n)}$

$\sum_1 \frac{(-1)^{n-1}}{(n-|z|)}$

2) find the convergence radius R of the series $\sum_1 (3 + (-1)^n)^n z^n$

2) fives the development in series of $\frac{1}{1+z}$

1) donner série de Laurent de $(z-1)cos(\frac{1}{z+2})$

1) g Which gives for

$$\Gamma(z) = \frac{\eta(z)}{(1 - 2^{1-z})2\sin(\pi z)\,\eta(1-z)}$$

1) show the order is unimportant in $\sum \sum \frac{1}{np^z}$ i.e.:

$$\sum_n \sum_p \frac{1}{np^z} = \sum_p \sum_n \frac{1}{np^z}$$

1) show that the function

$$log\xi(z) - \sum \log(1 - \frac{z}{\rho})$$

is even.

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The function $\Pi(x)$

with the aide of these auxiliairy principes, nous pouvons maintenant déterminer le nombre des nombres premiers qui sont inférieur à x.

This was the aim of **Riemann** article, bu it is not sure that **Riemann** knew the importance of his hypothèse qu'il considère comme ' *superflu pour mon but*', for it has big consequences on the distribution of prime numbers as said by specialists, we quote **Bombieri**[] :

Even a sole exception to the Riemann hypothesis aurait des énormément de étrange conséquences sur la distribution des nombre premiers.. si hypothèse de Riemann tourne être fausse il auraient de grand oscillations dan la distribution des premiers.

Dans un orchestre cela aurait etre un instrument qui fait les autres instrument- une situation esthétiquement

Soit $F(x)$ ce nombre lorsque x n'est pas exactement égal à un nombre premier, et soit $F(x)$ ce nombre augmenté de $\frac{1}{2}$, lorsque x est premier, de telle sorte que, pour une valeur x pour laquelle $F(x)$ varie par un saut brusque, on ait

$$F(x) = \frac{F(x+0) + F(x-0)}{2}$$

ce nombre $F(x)$ est noté aujourd'hui $\pi(x)$ (notation due à Landau) . La fonction $\pi(x)$ est une fonction discontinue aux points $x = p_i$, plus précisément elle est une fonction *continue par morceaux*.



Fig. the function, as a step function.

Riemann veut dire que

$$\pi(x) \quad = \text{si } x \neq p$$

And

$$\pi(x) + \frac{1}{2} \quad = \text{si } x = p$$

$\pi(x+)$ means the value at the right of de x et $\pi(x-)$ means the value at left of x. c'est this is a defintion by his teacher Gustave **Dirichlet** in the problem the convergence Fourier series

$$f(x) = a + \sum$$

the definition of $\pi(x)$ becomes then

$$\pi(x) = \frac{\pi(x+) + \pi(x-)}{2}$$

This Dirichlet has to the **Fourier** series he

$$f_n(x) = \int \frac{f(x+) + f(x-)}{2}$$

So in order that the converge it must verify the condition

$$f(x) = \frac{f(x+) + f(x-)}{2}$$

At the point of discontinuities

If now the expression

$$\boldsymbol{log\zeta(s) = \sum log(1 - p^{-s}) = \sum p^{-s} + \frac{1}{2}\sum p^{-2s} + \frac{1}{3}\sum p^{-2s} + \ldots}$$

when n=∞

if we take the logarithm in the **Eulerian** product

$$log\zeta(z) = log\prod(1 - \frac{1}{p})^{-1}$$

and we have the development of $log(1 - \varepsilon)$ where $\varepsilon = p^{-s}$, according to the known result of **Newton**

$$log(1 + x) = \sum_{n\geq 1}^{\infty} \frac{x^n}{n}(-1)^{n+1}$$

We find

$$\sum log(1 - p^{-s}) = \sum p^{-s} + \frac{1}{2}\sum p^{-2s} + \frac{1}{3}\sum p^{-2s} + \cdots$$

On remplace $\boldsymbol{p^{-s}}$ ***par*** $\boldsymbol{\int_p^{\infty} x^{-s-1}dx}$ **,** $\boldsymbol{p^{-s2}}$ **par** $\boldsymbol{s\int_{p^2}^{\infty} x^{-s-1}dx}$ **...**

on obtient

$$\boldsymbol{\frac{log\zeta(s)}{s} = \int_1^{\infty} f(x)x^{-s-1}dx}$$

where we désignate by $f(x)$ the expression

$$\boldsymbol{F(x)+\frac{1}{2}F(x^{\frac{1}{2}})+\frac{1}{3}F(x^{\frac{1}{3}})+\cdots}$$

He remplace an algébraic expresion by an integrale for if we calcule

$$z\int_{p}^{\infty}x^{-z-1}dx=[-x^{-z}]_{p}^{+\infty}$$

We find

$$z\int_{p}^{\infty}x^{-z-1}dx=\lim_{x\to+\infty}[-x^{-z}+p^{-z}]=p^{-z}$$

The function noted by **Riemann** $f(x)$ is today noted $\Pi(x)$

$$\Pi(x)=\pi(x)+\frac{1}{2}\pi(x^{\frac{1}{2}})+\frac{1}{3}\pi(x^{\frac{1}{3}})+\cdots$$

It has been used for the first time in a different variant by **Tchebychev** in 1852 when he settled **Bertrand**'s conjecture. In fact in his posthumous paper sheets that

remained at the university of Göttingen, **Riemann** had the intention to correspond with **Tchebychev**[29] .

in his proof of the **Bertrand** postulate, that there is always a prime in the interval $[n, 2n]$, **Tchebychev** introduced two new functions , namely

$$\theta(x) = \sum_{p \leq x} log(p)$$

and

$$\psi(x) = log(\prod_{1}^{n} p_i)$$

with the relation

$$\theta(x) + \theta(x)^{1/2} + \theta(x)^{1/3} \ldots = \psi(x)$$

which in modern

$$\psi(x) = \sum_{n \leq x} \Lambda(\mathrm{n})$$

we have also

$$-\frac{\zeta'(z)}{\zeta(z)} = \sum_{p=1} \Lambda(n) n^{-s}$$

[29] See **Edwards** p.4-5.

if $\pi(x)$ is defined as

$$\pi(x) = \sum_{p \leq x} 1$$

the function

$$\Pi(x) = \sum_{p^n \leq x} 1$$

we have then

$$\Pi(x) = \pi(x) + \frac{1}{2}\pi(x^{\frac{1}{2}}) + \frac{1}{3}\pi(x^{\frac{1}{3}}) + \cdots$$

to not that the sequence is finite when $x^{\frac{1}{n}} < 2$.

We have

$$log\zeta(z) = \sum \int_p^{\infty} x^{-z-1} dx + \frac{1}{2}\sum \int_p^{\infty} x^{-z-1} dx + \frac{1}{3}\sum \int_p^{\infty} x^{-z-1} dx \ldots$$

That gives

$$\frac{log\zeta(z)}{z} = \int_0^{\infty} \Pi(x)x^{-z-1}dx$$

This equation a lieu for all the complex values $a + ib$ of s when $a > 1$.

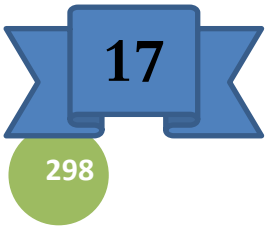
The equation

$$\frac{log\zeta(z)}{z} = \int_0^{\infty} \Pi(x)x^{-z-1}dx$$

Est valide pour $R(z) > 1$ ce qui veut dire que l'intégrale $\int_0^{\infty} \Pi(x)x^{-z-1}dx$ est convergente lorsque la réelle $R(z) > 1$.

Exercises

1) calculate :
i) $\Pi(0)$
ii) $\Pi(1)$
iii) $\Pi(10)$
1) find the convergence des séries
$\int_0^{\infty} \Pi(x)x^{-z-1}dx$
$\int_0^{\infty} \Pi(x)x^{-z-1}dx$
2) trouver le rayon de convergence de R
2) donner la
2) donner série



Fourier analysis

in analysis, the theory **Fourier** series and integral transform are of central importance and since Riemann has some work on this domain he used it to his results . So whe student is draticalt the veru ***Poisson*** *summation formula* without knowning why

$$\sum \hat{f} = \sum f$$

The F is and is and at the time of there no proof of this and

Mais lorsque, sous ces hypothèses, l'équation suivante

$$\boldsymbol{g(x) = \int_0^{\infty} h(x)x^{-s}dlogx}$$

a lieu, l'on peut, à l'aide du théorème de Fourier, exprimer la fonction $\boldsymbol{h}$ par la fonction $\boldsymbol{g}$.

Riemann veut dire par $théorème\ de\ Fourier$ l'opération qui s'appelle la *transformation de Fourier*: Fourier used his idea in 1807 when he to solve the heat eaution:

Let f and g deux fonctions some conditions that will be precised, then if

$$g(x) = \int_{-\infty}^{+\infty} f(t)e^{-i2\pi xt}dt$$

then

$$f(x) = \int_{-\infty}^{+\infty} g(t)e^{i2\pi xt}dt$$

In Riemann let f et g deux fonctions telles que

$$g(x) = \int_0^{\infty} f(x)x^{-s}dx$$

Alors on peut sortir f de l'intégrale sous certaines conditions

$$f(x) = \int_0^{\infty} g(x)x^{-s}dx$$

the **Fourier** transform finds very normally origin with the **Fourier** series : if $f(x)$ is par une série de Fourier on a

$$f(x) = \sum_{n=0}^{\infty} a_n cos(nx) + b_n sin(nx)$$

Alors les coefficients sont par

$$a_n = \int_0^{2\pi} f(x)\cos(nx)dx$$

but this is not always valable , there are some condition on the pair (f , g) d certaines de croissance and of convergnece known as Dirichlet conditions:

Le premier à utiliser le complexe dans l'analyse de Fourier était Poisson. On voit que **Riemann** a écrit $dlogx = \frac{dx}{x}$ alors $x^{-z}dlogx$ est le même que $x^{-z-1}dx$.

This equation, when $h(x)$ is real and also

$$\boldsymbol{g(a+ib) = g_1(a) + ig_2(b)}$$

Se décompose en les deux suivantes

$$\boldsymbol{g_1(b) = \int_0^{\infty} h(x)x^{-a}cos(blogx)dlogx}$$

$$\boldsymbol{ig_2(b) = -i\int_0^{\infty} h(x)x^{-a}sin(blogx)dlogx}$$

On a à condition que la fonction h est réelle, c'est à dire ne contient pas le nombre imaginaire i, on a alors

$$g(a+ib) = \int_0^\infty h(x)(a+ib)^{-z}dx$$

but

$$(a+ib)^{-z} = e^{log(a+ib)^{-z}} = a^x[\,cos(blogx) + i\,sin\,(blogx)]$$

that gives

$$g_1(b) = \int_0^\infty h(x)a^x cos(blogx)dx$$

et

$$ig_2(b) = -i\int_0^\infty h(x)a^x sin(blogx)dx$$

ou de meme

$$g(a+ib) = \int_0^\infty h(x)a^x cos(blogx)dx - i\int_0^\infty h(x)a^x sin(blogx)dx$$

Lorsque l'on multiplie les deux équations par

$$\boldsymbol{[cos(blogy) + isin(blogy)]db,}$$

et que l'on intègre de $-\infty$ à $+\infty$, l'on obtient , en vertu du théorème de Fourier , dans les second membres $\pi h(y)y^{-\alpha}$, et , par conséquent, en ajoutant les deux équations et multipliant par iy^{α} , on a

$$2\pi i h(y) = \int_{a-\infty i}^{a+\infty i} g(x)y^s ds,$$

où l'intégration doit être prise de telle sorte que la partie réelle de s reste constante.

Riemann ici la transformation de Fourier, par l'utilisation de l'astuce suivant : il multiplie l'équation de g_1 et ig_2 par le puis intégrer pour sortit la fonction h de de de l'intégrale et g de car on a

$$\frac{1}{2\pi}\int_0^{\infty}(\int_0^{\infty} h(x)x^{-s}\, dlogx)dlogx = h(y)$$

alors

$$[h(x)sin(blogx)dx]g_1(x) = [\cos(blogx) + sin(blogx)dx]\int_0^{\infty} h(x)x^{-s}dlogx$$

and

$$[h(x)sin(blogx)dx]g_2(x) = -i[\cos(blogx) + sin(blogx)dx]\int_0^{\infty} h(x)x^{-s}dlogx$$

Puis on intègre de 0 à –

$$\int_{a-i\infty}^{a+i\infty} h(x)sin(blogx)dx = \int_{-\infty}^{+\infty} h(x)sin(blogx)dx \int_0^{\infty} h(x)x^{-s}dlogx$$

Il tire

$$2\pi i h(y) = \int_{a-\infty i}^{a+\infty i} g(x) y^z dz,$$

or

$$h(y) = \frac{1}{2\pi i} \int_{a-\infty i}^{a+\infty i} g(x) y^z dz$$

lui-même a en 185 une des eri e de **Fourier**. Mais la transformation a sous des conditions. Les éditeurs des Œuvres de **Riemann** ont critisezed the raisonning of **Riemann** de cette opération

Cette intégrale représente, pour une valeur y pour laquelle a lieu une variation par saut brusque de la fonction, la valeur moyenne des valeurs de la fonction h de chaque côté du saut. Avec les modes de détermination exposés ci-dessus, la fonction $f(x)$ possède cette même propriété, et l'on a donc, d'une manière toute générale

$$\boldsymbol{f(y) = \frac{1}{2\pi i} \int_{a-\infty i}^{a+\infty i} \frac{log\zeta(s)}{s} y^s ds}$$

Le saut brusque par Riemann vient du fait que $\Pi(x)$, comme la fonction $\pi(x)$, est une fonction discontinue par morceaux

Alors de l'équation

$$log\zeta(z) = \int_{a-\infty i}^{a+\infty i} \frac{\Pi(x)}{z} x^z dz$$

On tire selon la transformation de Fourier

$$\Pi(x) = \frac{1}{i2\pi} \int_{a-\infty i}^{a+\infty i} \frac{log\zeta(z)}{z} x^z dz$$

today this special sort of transformation is known under the name of ***Mellin transform*** , was rediscovered by **Cahen** as we have seen, then by Finnish mathematician **Mellin**(1854-1933) in 1899 and took his name.

Mellin transform

From a **Laplace** transform that we know from differential equations if

$$L(f)(z) = \int gx^z dz$$

then

$$L(f)x) = \int_{ai}^{\infty i} gx^z dz$$

This can easily get from **Fourier** transform as follows :

We put in the Fourier transform $x =$ and $t =$ then we have:

Alors de l'équation

$$M(f)(z) = \int_{a-\infty i}^{a+\infty i} gx^z dz$$

On tire selon la transform

$$\mathrm{g(x)} = \frac{1}{2\pi} \int_{a-\infty i}^{a+\infty i} fx^z dz$$

for the function Γ

$$e^{x} = \frac{1}{2\pi i}\int_{a-\infty i}^{a+\infty i} \Gamma(z)x^{-z}dz$$

On peut maintenant substituer à $log\zeta(s)$ l'expression trouvée précédemment

$$\frac{s}{2}\log\pi - \log(s-1) - \log\Pi\left(\frac{s}{2}\right) + \sum_{\alpha} log\left(1+\frac{(s-\frac{1}{2})^2}{\alpha^2}\right) + log\xi(0)$$

Mais les intégrales de chaque terme de cette expression, prises jusqu'à l'infini, ne convergent pas; il sera donc convenable de transformer l'équation précédente à l'aide d'une intégration par partie

$$f(y) = \frac{1}{2\pi i}\frac{1}{logx}\int_{a-\infty i}^{a+\infty i} \frac{d[\frac{log\zeta(s)}{s}]}{ds}x^{s}ds$$

Puisque comme on a trouvé

$$\xi(z) = z(z-1)\pi^{-\frac{z}{2}}\,\zeta\,(z)\,\Gamma(\tfrac{z}{2})$$

et

$$\xi(z) = \xi(0)\prod\ (1-\tfrac{z}{\gamma_i})$$

en prenant le logarithme des deux équations

$$logz + log(z-1) - \tfrac{z}{2}log\pi + log\,\zeta\,(z) + log\,\Gamma(\tfrac{z}{2}) = log\xi(0) + \sum\ log(1-\tfrac{z}{2})$$

ce qui donne

$$log\ \zeta\ (z) = \ log\xi(0) + \sum_i log(1 - \frac{z}{\gamma_i})\ - logz - log(z-1) + \frac{z}{2} log\pi\ - log\ \Gamma(\frac{z}{2})$$

Une équation de six termes qui va être substituée dans l'intégrale

$$\frac{1}{i2\pi} \int_{a-\infty i}^{a+\infty i} \frac{log\zeta(z)}{z} x^z dz$$

obviousely inthe this substitution there is a *convergence problem* , soit la convergence de la série infinie ou la convergence des intégrales impropres.

Fig. integration in the complex plan $a > 1$

Une intégration par partie si f et g deux fonction

$$\int_b^a f dg\ = [fg]_b^a - \int_b^a g df$$

In our case

$$f = \frac{log\zeta(z)}{z} \quad et \quad dg = x^z dz$$

which gives

$$df = \frac{1 - log\zeta(z)}{z^2} \quad et \quad g = \frac{x^z}{logx}$$

Ce qui donne

$$\int_{a-\infty i}^{a+\infty i} \frac{log\zeta(z)}{z} x^z dz = [fg]_{a-\infty i}^{a+\infty i} - \int_{a-\infty i}^{a+\infty i} \frac{x^z}{logx} \frac{1 - log\zeta(z)}{z^2} dz$$

or

$$\Pi(x) = -\frac{1}{2i\pi logx} \int_{a-\infty i}^{a+\infty i} \frac{x^z}{} \frac{1 - log\zeta(z)}{z^2} dz$$

As

$$-\boldsymbol{log\Pi}(\tfrac{s}{2}) = \boldsymbol{lim}\,[\textstyle\sum_{n=1}^{n=m} \boldsymbol{log}\left(\mathbf{1} + \frac{s}{2n}\right) - \frac{s}{2}\boldsymbol{logm}]$$

For $m = \infty$, et que , then

$$-\frac{\boldsymbol{d\frac{1}{s}log\Pi(\frac{s}{2})}}{\boldsymbol{ds}} = \sum_{\mathbf{1}}^{\infty} \frac{\boldsymbol{d\frac{1}{s}log\Pi(1 + \frac{s}{2n})}}{\boldsymbol{ds}}$$

Cette est une de log la fonction gamma on a

$$\Gamma(\frac{z}{2}) = \lim_{n\to+\infty} \frac{n^{\frac{z}{2}} n!}{\frac{z}{2}\left(\frac{z}{2}+1\right)\ldots\left(\frac{z}{2}+n\right)}$$

then

$$-\log\Gamma(\tfrac{z}{2}) = [\textstyle\sum_{n=1}^{n=m} \log\left(1+\frac{s}{2n}\right) - \frac{s}{2}\log n]$$

on entre le logarithme car log est une fonction continue et en divisant les deux membres par z, par dérivation on trouve

$$-\frac{d\frac{1}{z}log\Gamma(\frac{z}{2})}{dz} = \sum_n \frac{d\frac{1}{z}log\Gamma(\frac{z}{2})}{dz}$$

for we know that

$$-\frac{d\frac{1}{z}\times\frac{z}{2}logm}{dz} = 0$$

Ce qui donne

tous les termes de l'expression de $f(x)$, à l'exception de

$$\frac{1}{2\pi i}\frac{1}{logx}\int_{a-\infty i}^{a+\infty i}\frac{1}{s^2}log\xi(0)x^s ds = log\xi(0),$$

prennent alors la forme

$$\pm\frac{1}{2\pi i logx}\int_{a-\infty i}^{a+\infty i}\frac{d[\frac{1}{s}log(1-\frac{s}{\beta})]}{ds}x^s ds$$

On a calculé ces intégrales définies des 6 termes de $log\ \zeta\ (z)$:

1) $\int_{a-i\infty}^{a+i\infty} \frac{d[log\xi(0)/z]}{dz} x^z dz = log\xi(0) \times \int_{a-i\infty}^{a+i\infty} d\frac{1}{zdz} x^z dz$

$$= log\xi(0) \times \int_{a-i\infty}^{a+i\infty} \frac{1}{z^2} x^z dz$$

$$\frac{d1}{dz} \int_{a-i\infty}^{a+i\infty} zdz =$$

On a $\frac{d[log\zeta]}{dz} = \frac{d[log\zeta]}{s}$

2) $\frac{1}{dz} \int_{a-i\infty}^{a+i\infty} \frac{d\sum_i log(1-\frac{z}{\gamma_i})}{dz} x^z dz = \frac{d1}{dz} \int_{a-i\infty}^{a+i\infty} \frac{d\frac{1}{z}}{dz} x^z dz$

$$= \frac{d1}{dz} \int_{a-i\infty}^{a+i\infty} \frac{1}{z^2} x^z dz$$

$$\frac{d1}{dz} \int_{a-i\infty}^{a+i\infty} zdz =$$

3) $\frac{1}{dz} \int_{a-i\infty}^{a+i\infty} \frac{d[log\xi(0)]}{dz} x^z dz = \frac{d1}{dz} \int_{a-i\infty}^{a+i\infty} \frac{d\frac{1}{z}}{dz} x^z dz$

$$= \frac{d1}{dz} \int_{a-i\infty}^{a+i\infty} \frac{1}{z^2} x^z dz$$

$$\frac{d1}{dz} \int_{a-i\infty}^{a+i\infty} zdz =$$

4) $\frac{1}{dz} \int_{a-i\infty}^{a+i\infty} \frac{d[log\xi(0)]}{dz} x^z dz = \frac{d1}{dz} \int_{a-i\infty}^{a+i\infty} \frac{d\frac{1}{z}}{dz} x^z dz$

$$= \frac{d1}{dz} \int_{a-i\infty}^{a+i\infty} \frac{1}{z^2} x^z dz$$

$$\frac{d1}{dz}\int_{a-i\infty}^{a+i\infty} zdz =$$

5) $\frac{1}{dz}\int_{a-i\infty}^{a+i\infty}\frac{d[log\xi(0)]}{dz}x^z dz = \frac{d1}{dz}\int_{a-i\infty}^{a+i\infty}\frac{d\frac{1}{z}}{dz}x^z dz$

$$= \frac{d1}{dz}\int_{a-i\infty}^{a+i\infty}\frac{1}{z^2}x^z dz$$

$$\frac{d1}{dz}\int_{a-i\infty}^{a+i\infty} zdz =$$

6) $\frac{1}{dz}\int_{a-i\infty}^{a+i\infty}\frac{d[log\xi(0)]}{dz}x^z dz = \frac{d1}{dz}\int_{a-i\infty}^{a+i\infty}\frac{d\frac{1}{z}}{dz}x^z dz$

$$= \frac{d1}{dz}\int_{a-i\infty}^{a+i\infty}\frac{1}{z^2}x^z dz$$

$$\frac{d1}{dz}\int_{a-i\infty}^{a+i\infty} zdz =$$

Mais on a maintenant

$$\frac{d[\frac{1}{s}log(1-\frac{s}{\beta})]}{d\beta} = \frac{1}{(\beta-s)\beta}$$

et, lorsque la partie réelle de s est plus grande que la partie réelle de β,

$$-\frac{1}{2\pi i}\int_{a-\infty i}^{a+\infty i}\frac{x^s ds}{(\beta-s)\beta} = \frac{x^\beta}{\beta} = \int_{\infty}^{x} x^{\beta-1}dx$$

ou bien

$$\int_{0}^{x} x^{\beta-1}dx$$

selon que le partie réelle de β est négative ou positive.

Par dérivation on a

$$\frac{d[\frac{1}{z}log(1-\frac{z}{\beta})]}{d\beta} = \frac{1}{z} \times \frac{d[log(1-\frac{z}{\beta})]}{d\beta} = \frac{1}{z} \times \frac{\frac{z}{\beta^2}}{\frac{\beta-z}{\beta}} = \frac{1}{(\beta-z)\beta}$$

Ce qui donne l'intégrale

$$\int_{a-\infty i}^{a+\infty i} \frac{x^z}{(\beta - z)\beta} dz$$

Le nombre β introduit ici est un nombre complexe comme z, alors pour trouver la limite

-le terme $log\xi(0)$ donne $log\frac{1}{2}$, Riemann a fait un erreur en évaluant cette constante.

-le terme $-logz$ donne

-le terme $-log(z-1)$ donne

-le terme $-log\,\Gamma(\frac{z}{2})$ donne

-le terme $\frac{z}{2}log\pi$ donne

-le terme $\sum_i log(1-\frac{z}{\gamma_i})$ donne $\sum_i Li(x^\rho)$

Comme suit :

L'intégrale $\int_{a-\infty i}^{a+\infty i} \frac{d[\frac{1}{z}log(1-\frac{z}{\beta})]}{dz} x^z dz$ converge car

Pour calculer les intégrales contenant le terme $\int_{a-\infty i}^{a+\infty i} \frac{x^z}{(\beta-z)\beta} dz$ les mathématiciens utilisaient une méthode de changement de variable calculant l'intégrale défini suivant

$$\int_{a-\infty i}^{a+\infty i} \frac{x^z}{z} dz$$

où est le chemin on peut de lorsque $\Re(z)$ tend vers ∞.

Pour ce problème vue de la présence de x^z rend le calcul difficile, on peut utiliser la transformation de Fourier utilisée auparavant on a

1) when $\Re(z) > \Re(\beta)$.

2)

3)

we have then, in the first case,

$$\pm \frac{1}{2\pi i} \frac{1}{\log x} \int_{a-\infty i}^{a+\infty i} \frac{d[\frac{1}{s} \log(1-\frac{s}{\beta})]}{ds} x^s ds$$

$$= \frac{1}{2\pi i} \int_{a-\infty i}^{a+\infty i} \frac{1}{s} \log(1-\frac{s}{\beta}) x^s ds = \int_{\infty}^{x} \frac{x^{\beta-1}}{\log x} dx$$

and in the second case

$$= \int_{0}^{x} \frac{x^{\beta-1}}{\log x} dx + const.$$

to evaluate the integral

$$\frac{1}{2i\pi \log x} \int_{a-\infty i}^{a+\infty i} \frac{d[\frac{1}{z} \log\left(1-\frac{z}{\beta}\right)]}{dz} x^z dz$$

on se sert d'abord de integration by parts

$$\int_{a-\infty i}^{a+\infty i} \frac{d[\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)]}{dz} x^z dz = [\frac{1}{z}\log\left(1-\frac{z}{\beta}\right)x^z]_{a-\infty i}^{a+\infty i} - \int_{a-\infty i}^{a+\infty i} \frac{1}{z}\log\left(1-\frac{z}{\beta}\right)(logx)x^z dz$$

which gives

$$\int_{a-\infty i}^{a+\infty i} \frac{1}{z}\log\left(1-\frac{z}{\beta}\right)x^z dz$$

whose evaluation gives the two integrals

$$\int_{\infty}^{x} \frac{x^{\beta-1}}{logx} dx \qquad or \qquad \int_{0}^{x} \frac{x^{\beta-1}}{logx} dx$$

according to the integration constant, ce qui va être expliquer dans ce qui suit.

In the first case, the integration constant may be determined by making the real part of β tend towards $-\infty$.

When we

Dans le second cas, l'intégration de 0 à x prend les valeurs qui diffèrent de $2\pi i$, lorsque l'intégrale relative à des valeurs complexes est prise dans le sens positif ou dans le sens négatif, et elle sera, prise dans ce dernier sens, infiniment petite lorsque le coefficient de i dans la valeur de β est égal à l'infiniment grand positif ; mais ce fait aura lieu, dans le premier cas, lorsque ce coefficient est égal à infiniment grand négatif.

Ceci nous enseigne comment $log(1-\frac{s}{\beta})$ doit être déterminé dans le premier membre de manière à faire disparaitre la constante d'intégration.

La fonction log est toujours accompagnée du problème de la détermination de la constante est.

Apres le changement de variable suivant

$$x^{\beta-1} = t$$

the result des deux derniers intégrales est

$$Li(x) = \int_2^x \frac{dt}{logt} +$$

Poisson summation formula:

Is tha to Fourier analysis its y Poisson in to the Fourier coefficients and the

$$\int_2^x \frac{dt}{logt} +$$

Exercises

1) extract the **Fourier** series for functions

i) $f = x$ on : $0 \leq x \leq 1$

ii) $f = -x$ on : $-1 \leq x \leq 1$

iii) $f = x + 1$ on : $0 \leq x \leq 1$

1) Calulate the **Fourier** transform of

$$f(x) = \begin{cases} 1 - 2x\,, & -1 \leq x \leq 0 \\ 1 + x, & 0 \leq x \leq 1 \\ 0, & elsewhere \end{cases}$$

1) from

$$2\xi(z) = \int_0^{+\infty} H(x)x^{-z}dt$$

Explain how to get $H(x)$.

2) trouver le rayon de convergence de R the series $\sum_1(3 + (-1)^n)^n z^n$

2) donner la série de $\frac{1}{1+z}$

2) donner série de Laurent de $(z-1)cos(\frac{1}{z+2})$

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The Prime Number Theorem

then **Riemann** arrives here to his final result qui which is the aim of his article:

En portant ces valeurs dans l'expression de $f(x)$, on obtient

$$f(x) = Li(x) - \sum_{\alpha}[Li(x^{\alpha}) + Li(x^{1-\alpha})] + \int_{x}^{\infty} \frac{dt}{t(t^2-1)logt} - log\xi(0), \text{ [3]}$$

où dans la série $\sum_{\alpha}$, on donnera à α pour valeur toutes les racines …

Riemann has discovered a very interesting relation between the distribution of prime numbers and the non-trivial zeros ρ of the function ζ which is he following equality

$$\Pi(x) = Li(x) - \sum_{Im\rho\geq 0}[Li(x^{\rho}) + Li(x^{1-\rho})] + \int_x^{\infty} \frac{dt}{t(t^2-1)logt} + log\xi(0)$$

Despite that the sum $\sum_{Im\rho\geq 0}[Li(x^{\rho}) + Li(x^{1-\rho})]$ implies complex quantities $\rho = a + i\gamma_n$, it always gives real number, avec les terms comme suit:

1) $\int_x^{\infty} \frac{dt}{t(t^2-1)logt}$ is a constant for

$$\int_x^{\infty} \frac{dt}{t(t^2-1)logt} \leq \int_x^{\infty} \frac{dt}{t^2 logt} = c$$

2) $\xi(0) = log2$, but **Riemann** made an error in its evaluation, and it is thought that the source of this error was the variable change $z = \frac{1}{2} + it$ that he made to get the function $\xi(\frac{1}{2} + it)$ (see **Edwards** p.31).

This formula has been considered by A.P. **Guinad** in 1942 and by **Weil**[30] in 1952 for function fileds.

…positives(or with positive real part) of the equation $\xi(\alpha) = 0$ by ranging them by increasing order.

Remains the problem that **Riemann** has not proved that the series

$$\sum_{Im\rho\geq 0} [Li(x^{\rho}) + Li(x^{1-\rho})]$$

[30] Sur les formules explicites de la théorie des nombres.

converges, it converges only conditionally (or it is a semi convergent series) which means any reorder of the roots can give another value , then the roots are conventionally ordered by increasing value norm

$$|\rho| = \left|\frac{1}{2} + i\alpha\right|$$

If not real then the root becomes

$$\rho = \frac{1}{2} + i(a + ib) = \frac{1}{2} - b + ia$$

if RH is false , **Riemann** means

$$\frac{1}{2} - b$$

but the sum meant that $\Re(\rho)$ is greater than 0, a fact that **Riemann** did not mention nor proved Edwards says (p.30):

The computation above assume $\Re(\rho) > 0$, but it has not been shown that this is true for all roots ρ. Although Hadamard later proved that there are no roots ρ on the line $\Re(\rho) = 0$, Riemann has not excluded this possibility and he is therefore not justified in ignoring the point as he does.

Then what is meant is when $x \longrightarrow +\infty$, it is not clear that

$$\Pi(x) + \sum_{Im\rho \geq 0} [Li(x^{\rho}) + Li(x^{1-\rho})] \sim \int_2^x \frac{dt}{logt}$$

Mathematicians after **Riemann** have tried to simplify his ideas and formulas. Von **Mangold** changed $\Pi(x)$ by $\psi(x)$ more accessible, defined a ew function

$$\Lambda = \begin{cases} logp\,, & if\ n = \\ 0, & otherwise \end{cases}$$

Ha proved the formula de **Riemann** under the form

von **Mangoldt** Theorem(1894) :

for $x > 1$ the formula

$$\psi(x) = x - \sum_{Im\rho \geq 0} \frac{x^{\rho}}{\rho} + \frac{1}{2} log\left(\frac{x^2}{x^2 - 1}\right) - log(2)$$

Is true

the work of von **Mangold** a facilitated the road to the proof of PNT , and by using **Tchebychev** function introduced , and he used a function namely

$$\psi(x) = \sum_{p \leq x} \log(p)$$

a task by both **Hadamard** and **Lavallée- Poussin** independently, this long sought result has now the form of

$$\psi(x) \sim x$$

If we take $\frac{1}{\zeta\,(z)}$ we find the series

$$\frac{1}{\zeta\,(z)} = \sum_{n\geq 1}^{\infty} \frac{\mu(n)}{n^z}$$

where the fonction de **Möbius** en théorie des nombres premiers

$$\mu(n) = \begin{cases} 1\ si\ n \\ 0\ sinon \end{cases}$$

the series is defined for $Re(z) > ½$. If one can prove that the series converges for the values $z = \frac{1}{2}$ this implies that the function has not roots in the half plane and the se **Riemann** hypothesis is proved ! this is what the Dutch mathematician **Stieltjes** thought he didi in 1885. His proof was not found.
after him , precisely en 1898, French mathematician Jacques **Hadamard** (18-19) and undependably of him, Belgium mathematician **La Vallée Poussin** (18-19) established a result about the distribution of prime numbers so searched since **Gauss** and **Legendre** :

Theorem(TNP):

$$\pi(x) \sim \frac{x}{log(x)} \sim \int_2^x \frac{dt}{log(t)}$$

Their proof consisted in showing that the function ζ does not vanish on the vertical line $\sigma = \Re(z) = 1$. In his article **Hadamard** wrote:

"…***Stieltjes*** *avait démontré, conformément aux prévisions de* ***Riemann*** *que ces zéros sont tous de la forme* $\frac{1}{2} + it$ *(le nombre t étant réel), mais sa démonstration n'a jamais été publiée, et il n'a même pas été établie que la fonction* ζ *n'ait pas de zéros sur la droite* $\Re(z) = 1$. C'est *cette dernière conclusion que je me propose de démontrer* ".

One of the biggest consequences , which has the consequent that

$$\sum_{Im\rho \geq 0} \frac{x^{\rho}}{\rho} = a < +\infty$$

which gives from

$$\psi(x) = x - \log(2\pi) + \frac{1}{2} log\left(\frac{x^2}{x^2 - 1}\right) - \sum_{Im\rho \geq 0} \frac{x^{\rho}}{\rho}$$

Because if $\rho = \sigma + i\gamma$ and $\sigma < 1$

$$\sum_{Im\rho \geq 0} \left|\frac{x^{\rho}}{\rho}\right| = \sum_{Im\rho \geq 0} \frac{x^{\sigma}}{|\rho|} \longrightarrow 0$$

So we obtain readily

$$\psi(x) \sim x$$

As **Chebychev** showed that

$$\pi(x) \sim \int_2^x \frac{1}{\log(t)} dt \iff \psi(x) \sim x$$

then

Theorem(PNT):

$$\pi(x) \sim \int_2^x \frac{1}{\log(t)} dt$$

Relation avec $\Pi(x)$, on a aussi cette intéressante relation entre $\pi(x)$ et la fonction $\zeta(z)$

Theorem ()

$$\zeta(x) = \int_2^x \frac{\pi(t)dt}{\log(t)}$$

In 1903 **Landau** gave a new proof of PNT without the use of the functional equation **Hadamard** theory of entire functions.

Exercises

1) calculate : $Li(x^{\rho})$

1)calculate $\Lambda(n)$ for :

1,2, 4, 10, 15,23

1) what do you think of the sum of the series

$$\sum_{Im\rho\geq 0} Li(x^{\rho}) + \sum_{Im\rho\geq 0} Li(x^{1-\rho})$$

1) show that $\sum_{Im\rho\geq 0}\frac{x^{\rho}}{\rho}$ diverges if $\sigma = \Re(\rho) > 1$.

1) show that

$$\frac{\zeta'(z)}{\zeta(z)} = -\sum_{p}\sum_{n\geq 1}(p^{-nz})logp$$

then give the relation between $\frac{\zeta'(z)}{\zeta(z)}$ and Λ.

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Order of the ζ function

Because of the complex variable, the nature de la function ζ is very complicated as an example **Hilbert** proved in

Theorem of **Hilbert**:

The function ζ cannot be the solution of a linear ordinary differential equation of coefficients.

this is also true for the Gamma function So mathematicians have sought to understand its behavior, one is how it behaves , when is grndeur when $|z| \to \infty$. at this **Tischmarch** declared(p6):

The fundamental problem which emerges from the attempt to determine the distribution of the prime numbers is : where are the zeros of the zeta-function? We also encounter the problem of the asymptotic behavior of $\zeta(s)$, as $t \longrightarrow \infty$, for given

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values of σ. The two problems are closely connected, and we cannot entirely separate them.

We have seen that regions without zeros that the location od the roots is connected to the growth of zeta sine it =0 infinitely of times and thus to the estimate of the error in PNT.

Les problèmes qui se posent naturellement :

—in what regions of the set $\mathbb{C}$ is the function ζ bounded ?
—if the function ζ is not bounded in a region of $\mathbb{C}$, what is its of , quel est son taux de croissance ?

We here some early result known in this domain of research:

Theorem:

The function ζ ne or mathematically:

$$\lim_{\substack{>\\ z\to 1}}(z-1)\zeta(z)=1$$

That is like the function z-1 at the neighborhood of 1.

par exemple d'étudier comment se la fonction

$$|\zeta(\sigma+it)|$$

on a deux : le nombre m qui est *le infinimum* de et le nombre M which is the *supremum*

Theorem

for real part σ we have :

$$m(\sigma) = \zeta(\sigma) \qquad \text{and} \qquad M(\sigma) = \frac{\zeta(2\sigma)}{\zeta(\sigma)}$$

The Lindelöf hypothesis

Si on dénote $\mu(\sigma)$ comme le nombre, ou de même hypothèse est équivalente à $\mu(\sigma) = \frac{1}{2}$

En 1908 **Lindelöf** the in : " *Some remarks on the increment of the* $\zeta(z)$ *function*". He began his paper by observing that

$$\prod_p (1 + p^{-z})^{-1} < |\zeta(\sigma + it)| \leq \prod_p (1 - p^{-z})^{-1}$$

He concluded that it is finite for $\geq 1 + \varepsilon$, then he about the behavior of the function $|\zeta(\sigma + it)|$ when $0 < \sigma$. By the functional equation

$$\Pi\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \Pi\left(\frac{1-s}{2}\right)\pi^{\frac{1-s}{2}}\zeta(1-s)$$

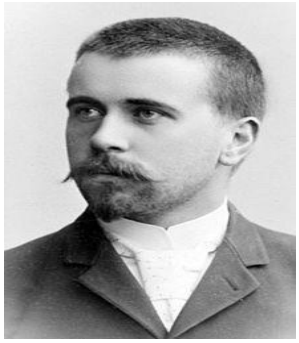
And

$$|\zeta(\sigma + it)| = |\zeta(\sigma - it)|$$

the approximation the gamma by **Stirling** formula

$$|\Gamma(\sigma + it)| = |t|e^{-|t|}$$

and by the **Euler**-**Maclaurin** formula he the



émet une hypotheses sur les de .

Lindelöf Hypotheses (1912):

let

$$[\zeta(\sigma + it)] \ll |t|^{\varepsilon}$$

which is equivalent to say that

$$Z(t) = o\,(t^{\varepsilon})$$

Or for every number $\varepsilon > 0$

$$\lim_{t \to +\infty} \frac{Z(t)}{t^{\varepsilon}} = 0$$

Lindelöf hypothesis has never been proved or disproved and we have a relation between it and the **Riemann** hypothesis

Littlewood (1912):

We have

$$\boldsymbol{RH} \Rightarrow \text{The Lindelöf hypothesis}$$

as it is very evident, the two conjectures are questioning about the behavior of the

function in the critical strip.

Backlund(1918):

We have

$$RH \Rightarrow \text{Lindelöf hypothesis}$$

The moments problem for $\zeta(z)$:

Since the zeta function vanishes on the critical strip and the roots are only on the critical line if the **Riemann** Hypothesis is true, mathematician tried to understand when it vanishes and thus they study the distribution of the roots. To the average (or mean) of a function f on an interval

$$\frac{1}{b-a}\int_a^b f(x)dx$$

The appearances of the roots is random and to study it the average to understand the

$$\frac{1}{2T}\int_{-T}^{T}\zeta(\frac{1}{2}+it)dt$$

As the function is symmetric we have only to take the interval $[0,T]$ and The $\zeta(\frac{1}{2}+it)$ is a complex number then we must the modulus

$$\frac{1}{T}\int_0^T\left|\zeta(\frac{1}{2}+it)\right|dt$$

The resulting integral is called usually a *moment*[31] , And we have

$$\left|\overline{f(z)}\right| = |f(z)|$$

The first result on moment to Hardy.

Exercises

1) find the values:

$$\left|\frac{1}{1-p^z}\right| , \quad \left|\frac{1}{1+p^z}\right| , \quad \left|\frac{p^z+1}{p^z}\right| , \quad \left|\frac{p^z-1}{p^z}\right|$$

1) show that

$$\prod_p \frac{p^\sigma}{p^\sigma+1} < \prod_p \frac{p^\sigma}{p^\sigma-1}$$

1) show that for $\sigma > 1$

$$\prod_p (1+p^{-z}) = \prod_p (1-p^{-2z}) \prod_p (1-p^{-z})^{-1}$$

1) show that : $|\zeta(\sigma+it)| = |\zeta(\sigma-it)|$

1) what is:

[31] To every time to invent names, mathematicians borrow names from physics so the word "*moment*" a word from physics and also it is also used in statistics , a moment in physics is $\sum fg$ or $\int fg$

$$\lim_{z \to -n} \frac{\Gamma(z)}{\zeta(z)} = 1$$

1) what is:

$$\lim_{z \to 1} \Gamma(z)\zeta(z) = 1$$

1) find the mean of the following real functions:

-$f(x) = x - 1$, on $[0,1]$.

- $f(x) = x^2 + 1$, on $[0,1]$.

-$f(x) = \frac{1}{x} - 2$, on $[0,1]$.

-$f(x) = \frac{1}{x^2} - 1$, on $[0,1]$.

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The Mobius inversion

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Conditional convergence:

We can then, by a more profound discussion of the function ξ , to prove easily, that when the terms are ordered as said above , the series

$$\sum_{\alpha}[Li\left(x^{\frac{1}{2}+\alpha i}\right)+Li\left(x^{\frac{1}{2}-\alpha i}\right)]\,logx\,,$$

converges towards the limit of the expression

$$\frac{1}{2\pi i}\int_{a-bi}^{a+bi}\frac{d\frac{1}{s}\sum[log(1-\frac{(s-\frac{1}{2})^2}{\alpha^2})]}{ds}x^s ds,$$

when the quantity b grows without finite limit.

we do not know where is this **Riemann** discussion of this propriety , and he surely

means that

$$\sum_{Im\rho\geq 0}[Li(x^{\rho})+Li(x^{1-\rho})]-\frac{1}{2\pi i}\int_{a-\infty i}^{a+\infty i}\frac{d\frac{1}{z}[\sum log(1-\frac{(z-\frac{1}{2})^2}{\beta})]}{dz}x^z dz$$

tends to 0 , when $b\longrightarrow+\infty$.

By one condition , if the he means that the infinite series involving the roots

$$\sum_{Im\rho\geq 0}[Li(x^{\rho})+Li(x^{1-\rho})]$$

converge only conditionally he explained :

but if we change the order of the series terms, we can obtain as a result any real number.

Riemann means that the series

$$\sum[Li\left(x^{\frac{1}{2}+\alpha i}\right)+Li\left(x^{\frac{1}{2}-\alpha i}\right)]$$

does not converge absolutely, which means that if we change the order of the zeros then we get whatever sum we want which a divergent series. This functions is an example par **Riemann** (p.22) himself by the infinite series

$$\sum (-1)^n \frac{1}{n}$$

that converge according to ***Leibnitz*** criteria , but it does not converge absolutely we have seen

$$\sum (-1)^n \frac{1}{n} = 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots$$

Sin on les termes paires et impaires on trouve

$$\sum (-1)^n \frac{1}{n} = -1 - \frac{1}{3} - \frac{1}{5} - \frac{1}{7} - \cdots + \frac{1}{2} + \frac{1}{4} + \frac{1}{6} \ldots$$

Ce a

by the aid of the function $f(x)$ l'on obtient $F(x)$ par inversion de la relation

$$f(x) = \sum \frac{1}{n} F^{(\frac{1}{n})}$$

ce qui donne l'équation

$$F(x) = \sum (-1)^{\mu} \frac{1}{m} f^{(x^{\frac{1}{m}})}$$

où m doit être remplace successivement par tous les nombres qui ne sont divisibles par aucun carrée excepté 1, et où μ désigne le nombre des facteurs premiers de m.

This operation which resembles the **Fourier** transform is called *de **Möbius** transform*, it was discovered independently by **Dedekind** and **Liouville** in 1834. It is like many

arithmetic functions that used much in number theory it on the ***Möbius*** function μ we seen before

$$\mu(x) = \sum f$$

Let $\Re(z) > 1$, we have seen before that

$$\frac{1}{\zeta(z)} = \sum_{1}^{+\infty} \frac{\mu(n)}{n^z}$$

Which give for $z = 1$

$$\frac{1}{\zeta(1)} = \sum_{1}^{+\infty} \frac{\mu(n)}{n} = 0$$

Since $\zeta(z)$ has a pole at $z = 1$ then $\frac{1}{\zeta(1)} = 0$. This result has been stated by **Euler** without proof and proved by von **Mangoldt** in 1897.

Given the equation

$$\Pi(x) = \pi(x) + \frac{1}{2}\pi\left(x^{\frac{1}{2}}\right) + \frac{1}{3}\pi\left(x^{\frac{1}{3}}\right) + \cdots$$

Of course **Riemann** is interested in the function $\pi(x)$ which counts the number of primes under x and not in the function $\Pi(x)$, so to get he made an inversion or a transform.
we get according to the **Möbius** transform

$$\pi(x) = \Pi(x) - \frac{1}{2}\Pi(x^{\frac{1}{2}}) - \frac{1}{3}\Pi(x^{\frac{1}{3}}) + \cdots$$

So **Riemann** made a good work , he indeed has found a link between the function $\pi(x)$ which represents the number of primes less than a real x and the function $Li(x)$ proposed by **Gauss** as only a conjecture to the number of such primes , and as not expected he with a strange complex function a relation still very explored.

Exercises

1) calculate : $Li\left(x^{\frac{1}{2}+2i}\right)$

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The order of the function $\pi(x)$

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Si on limite $\sum_{\alpha}$ à ***un nombre fini de termes, la dérivée de l'expression*** $f(x)$ ***c'est-à-dire, abstraction faite d'une partie qui décroit très rapidement lorsque*** x ***croit,***

$$\frac{1}{logx} - 2\sum_{\alpha}\frac{cos(\alpha logx)x^{-\frac{1}{2}}}{logx},$$

fournit une expression approchée pour la densité des entiers premiers $+$ ***la moitié de la densité des carrée,*** $+$ ***le tiers de celle des cubes,*** $+$ ***… des entiers premiers inferieurs à*** x***.***

On dans la formule que

$$\Pi(x) = Li(x) - \sum_{Im\rho\geq 0}[Li(x^{\rho}) + Li(x^{1-\rho})]$$

but

$$Li(x^{\rho}) = \frac{1}{logx^{\rho}} =$$

where

$$x^{\rho} = x^{\frac{1}{2}+it} = x^{\frac{1}{2}} \times x^{it} = x^{\frac{1}{2}} \times e^{itlogx} = x^{\frac{1}{2}}(cos(tlogx) + isin(tlogx))$$

Ce qui donne

$$\frac{1}{logx} - 2\sum_{\alpha} \frac{cos(\alpha logx)x^{-\frac{1}{2}}}{logx},$$

Ce qui donne

$$\frac{1}{logx} - 2\sum_{\alpha} \frac{cos(\alpha logx)x^{-\frac{1}{2}}}{logx} \approx \pi(x) + \frac{1}{2}\pi\left(x^{\frac{1}{2}}\right) + \frac{1}{3}\pi\left(x^{\frac{1}{3}}\right) + \cdots$$

La formule approche connu, $F(x) = Li(x)$, n'est, par conséquent, exacte qu'aux grandeurs près de l'ordre de $x^{\frac{1}{2}}$ et fournit une valeur un peu plus trop grande; car les termes non périodiques dans l'expression $F(x)$ sont, abstraction faite de grandeurs qui ne croissent pas indéfiniment avec x,

$$\boldsymbol{Li(x) - \frac{1}{2}Li(x) - \frac{1}{3}Li(x) - \frac{1}{5}Li(x) + \frac{1}{6}Li(x) - \frac{1}{7}Li(x) - \cdots}$$

since

$$\pi(x) = \Pi(x) - \frac{1}{2}\Pi(x^{\frac{1}{2}}) - \frac{1}{3}\Pi(x^{\frac{1}{3}}) + \cdots$$

and

$$\Pi(x) \approx Li(x)$$

alors approximatively

$$\pi(x) \approx Li(x) - \frac{1}{2}Li(x^{\frac{1}{2}}) - \frac{1}{3}Li(x^{\frac{1}{3}}) + \cdots$$

Lavalle-Poussin proved

$$\pi(x) \sim \int_2^x \frac{dt}{\log(t)} + O(xe^{-\Delta\sqrt{logx}})$$

where Δ is a real number > 0

Riemann says that the approximation $\pi(x) \sim \int_2^x \frac{dt}{\log(t)}$ conjectured by **Gauss**. les periodic terms in the expression of $\pi(x)$

$$\sum_{\alpha} \frac{cos(\alpha logx)x^{-\frac{1}{2}}}{logx}$$

in fact, ce ne sont pas périodique mais oscillatoires car un periodic a une periode il n'y a pas de période ici mai

In fact, the comparison, undertaken by Gauss and Goldschmidt, of $Li(x)$ with the number of primes less than x and pursued to $x = 3$ million has revealed that the number of primes, in the first hundred thousand, is always less than $Li(x)$ and that the difference in value, with some oscillations, increases as x increases.

The search for prime numbers and counting them i.e. to count the amount of prime numbers less than a quantity x, is ancient, before **Gauss** there many table of prime factors including clearly primes. One of the early table was a table to a book in algebra by a certain **Rohnius**(1656) published in Zurich, then extended by English John **Pell**(1610-1685) to 10.000. Anther **Felker** publishes a volume containing 408.000. Many efforts in the century like **Kulik**(1773-1863) calculated a table of 100.000.000 numbers factored but not published[32]. Here **Riemann** mention the

[32] See **Ore** (p.54).

problem of difference sign

$$Li(x) - \pi(x)$$

the comparison by **Gauss** when he computed the number of primes less than a real quantity x, and then by Carl W.B. **Goldschmidt** (1807-1851) student and assistant of **Gauss** at the of, who was a teacher of **Riemann** at Göttingen, he was also an astronomer

presented by function $\pi(x)$

already **Gauss** made a lot calculation and tables of primes when he was very young , they counted by chiliads[33].

x	Number of primes$< x$
500000	41538
10000000	78498
1500000	114155
2000000	148933
2500000	183072
3000000	216816

and tables of tables like tables of logarithms exits recently that **Dickson**, In the XX century **Lehmer** recently :

" *Large tables of primes are essentially obsolete now, since computers can generate primes afresh with sufficient rapidity for practical purposes*"[34].

Gauss to the number $x = 3.000.000$ and found that

[33] Chiliad means *thousand*, from Greek, so he integers in blocks of 1000 numbers..
[34] **Hardy** and **Wright**, of course it is the 2007 Edition, p. 12.

$$Li(x) > \pi(x)$$

In fact **Littlewood** proved the result:

Littlewood Theorem (1914)

the sign of the difference

$$Li(x) - \pi(x)$$

Changes infinitely many times.

it means that there are infinitely many points x_i such that $Li(x_i) > \pi(x_i)$ and infinitely many x_j such that $Li(x_j) < \pi(x_j)$ time, and is he showed that

$$\forall \varepsilon > 0, \exists x : \quad \pi(x) > Li(x) + x^{\frac{1}{2}-\varepsilon}$$

Littlewood found the existence of two positive constants α and β such that

$$Li(x) - \pi(x) > \alpha \frac{\sqrt{x} log log log(x)}{\log(x)}$$

and

$$Li(x) - \pi(x) < \beta \frac{\sqrt{x} log log log(x)}{\log(x)}$$

Mais la fréquence et la réunion plus denses par endroit des nombres premiers, si l'on peut s'exprimer ainsi, sous l'influence de des termes périodiques, avaient déjà attiré l'attention, lors du dénombrement des nombres premiers, sans que l'on eut

aperçu la possibilité d'établir une loi à ce sujet.

if one inspect the prime sequence, there are very near primes that qui constitute *blocks*

21, 32, 569, 45

And since

It would be interesting, in a new count of primes, to study the influence of every periodic term contained in the expression of the density of the prime numbers.

By periodic **Riemann** means that

$$Li(x^{\rho}) = \frac{1}{logx^{\rho}} =$$

Or

$$x^{\rho} = x^{\frac{1}{2}+it} = x^{\frac{1}{2}} \times x^{it} = x^{\frac{1}{2}} \times e^{itlogx} = x^{\frac{1}{2}}(cos(tlogx) + isin(tlogx))$$

Of course the function cos and sin are periodic but the terms are not periodic at all but oscillatory, as **Edwards** noted (p.305) :

"*Strictly speaking, the terms* $Li(x^{\frac{1}{2}+\alpha i})$ *are not periodic but merely oscillatory*".

Edwards in his manographe on the subsquent developpements following Riemann Riemann (2nd impression 2001) that

tant qu'il sache personne n'a d'étudier le problème de.

It seems that mathematicians ont trouvé pas un intérêt pour que and the big target was the proof of the $\boldsymbol{RH}$ qui est plus importante conséquences sur la distribution des nombres premiers.

Théorème de nombres premiers :

Toutes ces estimations veulent dire que

$$\lim_{\infty} \left| \pi(x) - \frac{x}{log(x)} \right| = 0$$

We use the absolute value for the difference is not always positive (as was demonstrated by **Littlewood** in 1914).

More regular than the behavior of $F(x)$, is the behavior of the function $f(x)$ which , very clearly in the first hundred, nearly coincides in mean with $Li(x) + log\xi(0)$.

the regularity here mean that the function $\Pi(x)$

$$\Pi(x) = Li(x) - \sum_{Im\rho \geq 0}[Li(x^{\rho}) + Li(x^{1-\rho})] + \int_{x}^{\infty} \frac{dt}{t(t^2-1)logt} + log\xi(0)$$

or noted $\mathrm{f}(x)$ by **Riemann** , is more regular than $\pi(x)$ or $\mathrm{F}(x)$

$$\pi(x) = \Pi(x) - \frac{1}{2}\Pi(x^{\frac{1}{2}}) - \frac{1}{3}\Pi(x^{\frac{1}{3}}) + \cdots$$

and we have seen that

$$\Pi(x) = \pi(x) + \frac{1}{2}\pi\left(x^{\frac{1}{2}}\right) + \frac{1}{3}\pi\left(x^{\frac{1}{3}}\right) + \cdots$$

And this that

$$\Pi(x) \sim Li(x) + log\xi(0).$$

Titshsmarsh p.2 :

Theorem ()

$$log\zeta(z) = \mathrm{z}\int_{2}^{\infty} \frac{\pi(t)dt}{\mathrm{t}(\mathrm{t}^{z}-1)}$$

Exercises

1) calculate $Li(x^{\rho})$ and $+Li(x^{1-\rho})$ for ρ.

2) montrer que $\sum_p \frac{1}{p^{\sigma}}$ converge pour $\sigma > 1$

2) montrer que $\sum_n \frac{1}{np^n}$ converge

2) montrer que $\sum_{p\leq x} \frac{1}{p} \sim \log(\log(x))$

2) montrer que $\sum_{p,n} \frac{1}{pn}$ converge

2) calculer $\cos(3\theta)$à l'aide $\cos(\theta)$

2) calculer $cos^2(\theta)$à l'aide $\cos(2\theta)$

2)calculer $\int_0^{\frac{\pi}{2}} sin^3(\theta)d\theta$ et $\int_0^{\frac{\pi}{2}} cos^3(\theta)d\theta$ par deux méthodes

2) montrer les égalités suivantes :

$\sin(i\theta) = i\sinh(\theta)$

$\sinh(i\theta) = i\sin(\theta)$

$\cos(i\theta) = \cosh(\theta)$

$\cosh(i\theta) = \cos(\theta)$

2) donner les séries entières de $\sinh(\theta)$ et $\cosh(\theta)$

2) montrer que $\cos(z) = cos(x)cosh(y) - isin(x)sinh(y)$

2) trouver $\frac{1}{\cos(z)}$ et $\frac{1}{\sin(z)}$ au de $log(z)$

2) calculer $cosh^2(z) + sinh^2(z)$

2) est ce que $f(z) = z + i$ le principe de réflexion

2) donner le théorème de Green-Riemann dans le plan complexe ($\mathbb{R}^2$)

2) montrer que la série $\zeta(z)$ converge pour $\sigma > 1$

2) montrer que la série $\zeta(\rho)$ =0 alors $\zeta(\bar{\rho})$ =0

2) intégrer $f(z) = z$ sur la courbe γ(t) $= 1 + (i-1)t$ avec $0 \leq t \leq 1$

2) intégrer $f(z) = z^2 + 1$ sur la cercle unité

III) Epilogue

More Riemann's works

Of a deteriorated health, **Riemann** had pain to do mathematics rightly, and despite that he continued his scientific works after 1859, we can sketch his afterwards articles :

—*Proof of the theorem that a single-valued periodic function of n variables cannot be more than 2n-fold periodic* (letter to **Weierstrass** in 1859), he mentioned in this letter a new formula for the zeta function that he could not simplify to be published.

—*On propagation of planar air waves with a finite amplitude of vibration* (Göttingen 1860).

—*On the minimal surfaces of aire for a given contour* (Göttingen 1860-61).

—*A contribution to the study of the motion of a homogenous fluid ellipsoid.* (Göttingen 1861).

—*On the attraction due to any homogenous right ellipsoid cylinder*(letter to **Betti** 1864).

—*On the vanishing of* θ —*functions*(Göttingen 1865).

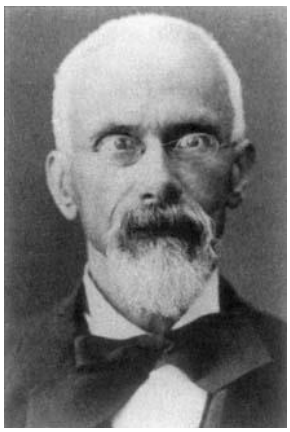
Illness

Riemann visited Paris in 1860 for a month. There he met many French mathematicians as **Serret**(18-1), **Hermite**(18-1), **Betrand**(18-1) , **Bouquet**(18-1), **Puiseux**(18-1), **Briot**(18-1).

Just after his marriage in 1863, **Riemann** fall sick with tuberculosis, the formidable illness. His friend **Eisenstein** succumbed to the disease prematurely in 1852 at the age of 29. **Riemann** advised by physicians to go to Italy for its adequate climate, so he made two trips to the country: the first in 18 and the second to Siliciae in 1865.

There he also met many persons of the Italian mathematical community , among them **Beltrami**(181-) and Enrico **Betti** (1823-1892), the latter whom he already met in Berlin at the autumn of 1858 with , Feleci **Casorati**(1835-1890), Francesco **Brioschi**(1828-1894) during their visit in Europe. **Riemann** send him a letter in 1864 on a mathematical problem. Italian **Gennocci**(18-1) wrote a report on **Riemann** paper on zeta function in 1860 in *Annali di Matematica pura e applicata.*

Betti translated **Riemann** Dissertation in 1859 and wrote an article on elliptic functions (1860). In the same year he wrote a report on **Riemann** paper about the propagation of planar waves. He also proposed him a post at the university of Pisa , which **Riemann,** because of his health, was obliged to decline. While in the Italian town Pisa, in 1864 .



Riemann wrote to **Betti** a letter about the problem in electrostatic, it was about the attraction of an homogenous right ellipsoid cylinder, and how to calculate the potential by **Dirichlet** method invoking the and **Dirichlet** problem

Discussions of **Riemann** and **Betti** about his ideas on elliptic integrals were at the beginning of algebraic topology, and have been developed afterwards by **Poincaré** and gave the **Betti** numbers as the **Riemann** about algebraic functions.

Death

The 5th of June of 1866, **Riemann** left for a second time with his wife for Italy where he died on the 20th of june at Verbania , a town situated at the bank of the Lac Majeur between Switzerland and Italy .



Fig.Laggio Magiorre

Linked to God and still fond of research, **Riemann** died at the age of flowers ,only God knows what he would give more to mathematics if he lived.


HIER RUHET IN GOTT
GEORG FRIEDRICH BERNHARD RIEMANN
PROFESSOR ZU GOETTINGEN
GEBOREN IN BRESELENZ DEN 17. SEPTEMBER 1826
GESTORBEN IN SELASCA DEN 20. JULI 1866
DENEN DIE GOTT LIEBEN MUESSEN
ALLE DINGE ZUM BESTEN DIENEN

Edited work

The work of **Riemann** has been edited and published mainly by his friend **Dedekind** and German mathematicien Henrich **Weber**(18-19) , the first editon was printed in 1876 , the second in 1892. The latter was enriched with posthumous fragments called

in German the *Nachlass*. In 19 edition German Max **Noether**(18-19) also added some notes .

Riemann has left also some mathematics courses which he was used to teach at Göttingen, they were a collection of notes form , they have been edited by **Hattendorff** , **Stahl**,... Other notes were preserved by German physicist Ernst **Abbe**(1840-1905) who studied in Gottingen(1859-1861). He also left many correspondence letters with his family and other mathematicians.

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These books tells anecdotes, historical circumstance of invention and discoveries.

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